

ANTI-CONCENTRATION AND HONEST ADAPTIVE CONFIDENCE BANDS

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ABSTRACT. Modern construction of uniform confidence bands for nonparametric densities (and other functions) often relies on the Smirnov-Bickel-Rosenblatt (SBR) condition; see e.g. Gine and Nickl (2010). This condition requires existence of a limit distribution of an extreme value type for a supremum of a studentized empirical process (equivalently, for a supremum of a Gaussian process with an equivalent covariance kernel). The principal contribution of this paper is to remove the need for SBR condition. We show that a weaker sufficient condition is the anticoncentration inequality for the supremum of the approximating Gaussian process, and we derive such an inequality under weak assumptions. Our new result shows that the supremum does not concentrate too fast around its expected value. We then apply this result to derive a Gaussian bootstrap procedure for constructing honest and adaptive confidence bands for nonparametric density estimators, completely avoiding the need for SBR condition. An essential advantage of our approach is that it applies even in those cases where the limit distribution does not exist (or is unknown). Furthermore, our approach provides an approximation to the exact finite sample distribution with an error that converges to zero at a fast, polynomial speed (with respect to the sample size). In sharp contrast, the Smirnov-Bickel-Rosenblatt approach provides an approximation with an error that converges to zero at a slow, logarithmic speed.

1. INTRODUCTION

Let X_1, \dots, X_n be i.i.d. random variables with common unknown density f on \mathbb{R}^d . We are interested in constructing confidence bands for f on a subset $\mathcal{X} \subset \mathbb{R}^d$ that are *honest* and *adaptive* to a given class \mathcal{F} of densities on \mathbb{R}^d . Typically, \mathcal{X} is a compact set on which f is bounded away from zero, and \mathcal{F} is a class of smooth densities such as a subset of a Hölder ball. A confidence band $C_n = C_n(X_1, \dots, X_n)$ is a family of random intervals

$$C_n := \{C_n(x) = [c_L(x), c_U(x)] : x \in \mathcal{X}\}$$

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that contains the graph of f on \mathcal{X} with a guaranteed probability. Following [19], a band C_n is said to be *asymptotically honest with level $\alpha \in (0, 1)$ for the class \mathcal{F}* if

$$\liminf_{n \rightarrow \infty} \inf_{f \in \mathcal{F}} P_f(f(x) \in C_n(x), \forall x \in \mathcal{X}) \geq 1 - \alpha.$$

Let $\hat{f}_n(\cdot, l)$ be a generic estimator of f with a smoothing parameter l , say bandwidth or resolution level, where l is chosen from a candidate set \mathcal{L}_n . Let $\hat{l}_n = \hat{l}_n(X_1, \dots, X_n)$ be a possibly data-dependent choice of l in \mathcal{L}_n . Denote by $\sigma_{n,f}(x, l)$ the standard deviation of $\sqrt{n}\hat{f}_n(x, l)$, i.e., $\sigma_{n,f}(x, l) := \sqrt{n \text{var}_f(\hat{f}_n(x, l))}$. Then, we consider a confidence band of the form

$$C_n(x) = \left[\hat{f}_n(x, \hat{l}_n) - c(\alpha) \frac{\sigma_{n,f}(x, \hat{l}_n)}{n^{1/2}}, \hat{f}_n(x, \hat{l}_n) + c(\alpha) \frac{\sigma_{n,f}(x, \hat{l}_n)}{n^{1/2}} \right] \quad (1.1)$$

where $c(\alpha)$ is a (possibly data-dependent) critical value determined to make the confidence band to have level α . Generally, $\sigma_{n,f}(x, l)$ is unknown and has to be replaced by an estimator.

A crucial point in construction of confidence bands is the computation of the critical value $c(\alpha)$. Assuming that $\sigma_{n,f}(x, l)$ is positive on $\mathcal{X} \times \mathcal{L}_n$, define the stochastic process

$$Z_{n,f}(v) := Z_{n,f}(x, l) := \frac{\sqrt{n}(\hat{f}_n(x, l) - E_f[\hat{f}_n(x, l)])}{\sigma_{n,f}(x, l)} \quad (1.2)$$

for $v = (x, l) \in \mathcal{X} \times \mathcal{L}_n =: \mathcal{V}_n$. We refer to $Z_{n,f}$ as a “studentized process”. If, for the sake of simplicity, the bias $|f(x) - E_f[\hat{f}_n(x, l)]|_{l=\hat{l}_n}$ is sufficiently small compared to $\sigma_{n,f}(x, \hat{l}_n)$, then

$$\begin{aligned} P_f\{f(x) \in C_n(x), \forall x \in \mathcal{X}\} &\approx P_f\left(\sup_{x \in \mathcal{X}} |Z_{n,f}(x, \hat{l}_n)| \leq c(\alpha)\right) \\ &\geq P_f\left(\sup_{v \in \mathcal{V}_n} |Z_{n,f}(v)| \leq c(\alpha)\right), \end{aligned}$$

so that the band (1.1) will be of level $\alpha \in (0, 1)$ by taking

$$c(\alpha) = (1 - \alpha)\text{-quantile of } \|Z_{n,f}\|_{\mathcal{V}_n} := \sup_{v \in \mathcal{V}_n} |Z_{n,f}(v)|, \quad (1.3)$$

which is, however, infeasible since the finite sample distribution of the process $Z_{n,f}$ is unknown. Instead, we estimate the $(1 - \alpha)$ -quantile of $\|Z_{n,f}\|_{\mathcal{V}_n}$.

Suppose that one can construct a sequence of random variables $W_{n,f}^0$ that are equal in distribution to the suprema of zero mean tight Gaussian processes $G_{n,f}$ indexed by \mathcal{V}_n with known or estimable covariance structure $(W_{n,f}^0 \stackrel{d}{=} \|G_{n,f}\|_{\mathcal{V}_n})$, and such that $\|Z_{n,f}\|_{\mathcal{V}_n}$ is close to $W_{n,f}^0$. Then, we may approximate the $(1 - \alpha)$ -quantile of $\|Z_{n,f}\|_{\mathcal{V}_n}$ by

$$c_{n,f}(\alpha) := (1 - \alpha)\text{-quantile of } \|G_{n,f}\|_{\mathcal{V}_n}.$$

Typically, one computes or approximates $c_{n,f}(\alpha)$ by one of two methods below.

1. Analytical method: derive analytically an approximated value of $c_{n,f}(\alpha)$, by using an explicit limit distribution or large deviation inequalities.
2. Simulation method: simulate the Gaussian process $G_{n,f}$ to compute $c_{n,f}(\alpha)$ numerically, by using, for example, a multiplier method.

The main purpose of the paper is to introduce a general approach to establishing the validity of so-constructed confidence bands. Importantly, our analysis does not rely on the existence of an explicit (continuous) limit distribution of any kind, which is a major difference from the previous literature. If, for some normalizing constants b_n and d_n , $d_n(\|G_{n,f}\|_{\mathcal{V}_n} - b_n)$ has a continuous limit distribution, the validity of confidence bands constructed would follow via the continuity of the limit distribution. For the density estimation problem, if \mathcal{L}_n is a singleton, i.e., the bandwidth or resolution level is chosen deterministically, the existence of such a continuous limit distribution, which is typically a Gumbel distribution, has been established for kernel density estimators and *some* wavelet density estimators [see 25, 1, 11, 14, 3, 4, 10]. We refer to the existence of the limit distribution as Smirnov-Bickel-Rosenblatt (SBR) condition. However, SBR condition has not been obtained for other density estimators such as those based on projection kernels with orthogonal polynomials and trigonometric functions. In addition, to guarantee the existence of a continuous limit distribution often requires more stringent regularity conditions than a Gaussian approximation itself. More importantly, if \mathcal{L}_n is not a singleton, which is typically the case when \hat{l}_n is data-dependent, and so the randomness of \hat{l}_n has to be taken into account, it is often hard to determine an exact limit behavior of $\|G_{n,f}\|_{\mathcal{V}_n}$.

We thus take a different route. Our key ingredient is the *anti-concentration* property of suprema of Gaussian processes. In studying the effect of approximation and estimation errors on the coverage probability, it is required to know how random variable $\|G_{n,f}\|_{\mathcal{V}_n} := \sup_{v \in \mathcal{V}_n} |G_{n,f}(v)|$ concentrates or “anti-concentrates” around, say, its $(1 - \alpha)$ -quantile. It is not difficult to see that $\|G_{n,f}\|_{\mathcal{V}_n}$ itself has a continuous distribution, so that *with keeping n fixed*, the probability that $\|G_{n,f}\|_{\mathcal{V}_n}$ falls into the interval with center $c_{n,f}(\alpha)$ and radius ϵ goes to 0 as $\epsilon \rightarrow 0$. However, what we need to know is the behavior of those probabilities when ϵ is n -dependent and $\epsilon = \epsilon_n \rightarrow 0$. In other words, bounding explicitly “anti-concentration” probabilities for suprema of Gaussian processes is desirable. We will first establish bounds on the Lévy concentration function (see Definition 2.1) for suprema of Gaussian processes and use these bounds to quantify the effect of approximation and estimation errors on the finite sample coverage probability.

1.1. Related references. Confidence bands in nonparametric estimation have been extensively studied in the literature. A classical approach, which

goes back to [25] and [1], is to use explicit limit distributions of normalized suprema of studentized processes. A “Smirnov-Bickel-Rosenblatt type limit theorem” combines Gaussian approximation techniques and extreme value theory for Gaussian processes. It was argued that the convergence to normal extremes is considerably slow despite that the Gaussian approximation is relatively fast [16]. To improve the finite sample coverage, bootstrap is often used in nonparametric estimation [see 5, 2]. However, to establish the validity of bootstrap confidence bands, they relied on the existence of continuous limit distributions of normalized suprema of original studentized processes. In the deconvolution density estimation problem, [20] considered confidence bands without using Gaussian approximation. In the current density estimation problem, their idea reads as bounding the deviation probability of $\|\hat{f}_n - E[\hat{f}_n(\cdot)]\|_\infty$ by using Talagrand’s [26] inequality and replacing the expected supremum by the Rademacher average. Such a construction is indeed general and applicable to many other problems, but is likely to be more conservative than our construction.

1.2. Organization of the paper. In the next section, we give a new anti-concentration inequality for suprema of Gaussian processes. Section 3 describes two new coupling inequalities. Together, Sections 2 and 3 provide powerful tools for proving validity of bootstrap methods to estimate distributions of suprema of empirical processes of VC type function classes. Section 4 contains theory of honest and adaptive confidence band construction under high level conditions. These conditions are easily satisfied both for convolution and projection kernel techniques under mild assumptions. Section 5 gives primitive sufficient conditions. Finally, all proofs are contained in the Appendix.

1.3. Notation. In what follows, constants $c, C, c_1, C_1, c_2, C_2, \dots$ are understood to be independent of n and to be *strictly* positive. The values of c and C may change at each appearance but constants $c_1, C_1, c_2, C_2, \dots$ are fixed. Throughout the paper, $E_n[\cdot]$ denotes the average over index $1 \leq i \leq n$, i.e., it simply abbreviates the notation $n^{-1} \sum_{i=1}^n [\cdot]$. E.g., $E_n[x_{ij}^2] = n^{-1} \sum_{i=1}^n x_{ij}^2$. Finally, for a function $\{Y(t) : t \in T\}$, $\|Y(t)\|_T$ denotes the supremum norm, i.e. $\|Y(t)\|_T := \sup_{t \in T} |Y(t)|$.

2. ANTI-CONCENTRATION OF SUPREMA OF GAUSSIAN PROCESSES

The main purpose of this section is to derive an upper bound on the *Lévy concentration function* for suprema of separable stochastic processes, where the terminology is adapted from [23]. Let (Ω, \mathcal{A}, P) be the underlying probability space.

Definition 2.1. Let $Y = (Y_t)_{t \in T}$ be a separable stochastic process indexed by a semimetric space T . For all $x \in \mathbb{R}$ and $\epsilon \geq 0$, let

$$p_{x,\epsilon}(Y) := P \left(\left| \sup_{t \in T} Y_t - x \right| \leq \epsilon \right). \quad (2.1)$$

Then the *Lévy concentration function* of $\sup_{t \in T} Y_t$ is defined for all $\epsilon \geq 0$ as

$$p_\epsilon(Y) := \sup_{x \in \mathbb{R}} p_{x,\epsilon}(Y). \quad (2.2)$$

Likewise, define $p_{x,\epsilon}(|Y|)$ by (2.1) with $\sup_{t \in T} Y_t$ replaced by $\sup_{t \in T} |Y_t|$ and define $p_\epsilon(|Y|)$ by (2.2) with $p_{x,\epsilon}(Y)$ replaced by $p_{x,\epsilon}(|Y|)$.

Let $X = (X_t)_{t \in T}$ be a separable Gaussian process indexed by a semimetric space T such that $E[X_t] = 0$ and $E[X_t^2] = 1$ for all $t \in T$. Assume that $\sup_{t \in T} X_t < \infty$ a.s. Our aim here is to obtain a qualitative bound on the concentration function $p_\epsilon(X)$. In a trivial example that T is a singleton, i.e., X is a real standard normal random variable, it is immediate to see that $p_\epsilon(X) \asymp \epsilon$ as $\epsilon \rightarrow 0$. A non-trivial case is that when T and X are indexed by $n = 1, 2, \dots$, i.e., $T = T_n$ and $X = X^n = (X_{n,t})_{t \in T_n}$, and the complexity of the set $\{X_{n,t} : t \in T_n\}$ (in $L^2(\Omega, \mathcal{A}, P)$) is increasing in n . In such a case, it is typically not known whether $\sup_{t \in T_n} X_{n,t}$ has a limiting distribution as $n \rightarrow \infty$ and therefore it is not trivial at all whether, for any sequence $\epsilon_n \rightarrow 0$, $p_{\epsilon_n}(X^n) \rightarrow 0$ as $n \rightarrow \infty$, which is in fact generally not true as Example 1 in [7] shows.

Theorem 2.1. *Let $X = (X_t)_{t \in T}$ be a separable Gaussian process indexed by a semimetric space T such that $E[X_t] = 0$ and $E[X_t^2] = 1$ for all $t \in T$. Assume that $\sup_{t \in T} X_t < \infty$ a.s. Then, $a(X) := E[\sup_{t \in T} X_t] \in [0, \infty)$ and*

$$p_\epsilon(X) \leq A\epsilon(a(X) \vee 1) \quad (2.3)$$

for all $\epsilon \geq 0$ and some absolute constant A .

The similar conclusion holds for the concentration function of $\sup_{t \in T} |X_t|$.

Corollary 2.1. *Let $X = (X_t)_{t \in T}$ be a separable Gaussian process indexed by a semimetric space T such that $E[X_t] = 0$ and $E[X_t^2] = 1$ for all $t \in T$. Assume that $\sup_{t \in T} X_t < \infty$ a.s. Then, $a(|X|) := E[\sup_{t \in T} |X_t|] \in [\sqrt{2/\pi}, \infty)$ and*

$$p_\epsilon(|X|) \leq A\epsilon a(|X|) \quad (2.4)$$

for all $\epsilon \geq 0$ and some absolute constant A .

We refer to (2.3) and (2.4) as anti-concentration inequalities because they show that suprema of separable Gaussian processes can not concentrate too fast. The proof of Theorem 2.1 and Corollary 2.1 follows by extending the results in [7] where we derived anti-concentration inequalities for maxima of Gaussian vectors. See Appendix for a detailed exposition.

3. COUPLING INEQUALITIES

The purpose of this section is to provide two new coupling inequalities that will be useful for the analysis of uniform confidence bands. The first inequality is concerned with suprema of empirical processes and is a direct corollary of Theorem 2.1 in [6]. The second inequality is concerned

with suprema of Gaussian multiplier processes and will be obtained from a Gaussian comparison theorem derived in [7].

Let X_1, \dots, X_n be i.i.d. random variables taking values in a measurable space (S, \mathcal{S}) . Let \mathcal{G} be a class of functions defined on S . We assume that \mathcal{G} is a separable class of measurable functions uniformly bounded by a constant b such that the covering numbers of \mathcal{G} satisfy

$$\sup_Q N(\mathcal{G}, L_2(Q), b\tau) \leq (a/\tau)^v, \quad 0 < \tau < 1$$

for some $a \geq e$ and $v \geq 1$ where the supremum is taken over all measures Q on (S, \mathcal{S}) . We refer to function classes with these properties as VC type with constants a and v and constant envelope b . Let σ^2 be a constant such that $\sup_{g \in \mathcal{G}} \mathbb{E}[g(X_i)^2] \leq \sigma^2 \leq b^2$. Define the empirical process

$$\mathbb{G}_n(g) := \frac{1}{\sqrt{n}} \sum_{i=1}^n (g(X_i) - \mathbb{E}[g(X_i)]), \quad g \in \mathcal{G},$$

and let $W_n := \|\mathbb{G}_n\|_{\mathcal{G}} := \sup_{g \in \mathcal{G}} |\mathbb{G}_n(g)|$ denote the supremum of the empirical process. Let $B = \{B(g) : g \in \mathcal{G}\}$ be a centered tight Gaussian process with covariance function

$$\mathbb{E}[B(g_1)B(g_2)] = \mathbb{E}[g_1(X_i)g_2(X_i)] - \mathbb{E}[g_1(X_i)]\mathbb{E}[g_2(X_i)]$$

for all $g_1, g_2 \in \mathcal{G}$. It is well known that such a process exists. Finally, for some sufficiently large but absolute constant A , denote

$$K_n := Av(\log n \vee \log(ab/\sigma)).$$

The following theorem shows that W_n can be well approximated by the supremum of the corresponding Gaussian process B under mild conditions on b , σ , and K_n .

Theorem 3.1. *Consider the setting specified above. Then for any $\gamma \in (0, 1)$ one can construct on an enriched probability space a random variable W^0 such that (i) $W^0 \stackrel{d}{=} \|B\|_{\mathcal{G}}$ and (ii)*

$$\mathbb{P} \left(|W_n - W^0| > \frac{bK_n}{\gamma^{1/2}n^{1/2}} + \frac{\sigma^{1/2}K_n^{3/4}}{\gamma^{1/2}n^{1/4}} + \frac{b^{1/3}\sigma^{2/3}K_n^{2/3}}{\gamma^{1/3}n^{1/6}} \right) \leq A \left(\gamma + \frac{\log n}{n} \right)$$

for some absolute constant A .

Comment 3.1. The main advantage of the coupling provided in this theorem in comparison with, say, Hungarian coupling [18], which can be used to derive a similar result, is that our coupling does not impose any side restrictions. In particular, it does not require bounded support of X and allows for point masses on the support. In addition, if the density of X exists, our coupling does not assume that this density is bounded away from zero on the support. Finally, our coupling does not assume that functions $g \in \mathcal{G}$ have bounded variation. See, for example, [22] for the construction of the Hungarian coupling and the use of aforementioned conditions.

Let ξ_1, \dots, ξ_n be i.i.d. $N(0, 1)$ random variables independent of $X_1^n := \{X_1, \dots, X_n\}$. Denote $\xi_1^n := \{\xi_1, \dots, \xi_n\}$. We assume that random variables $X_1, \dots, X_n, \xi_1, \dots, \xi_n$ are defined as coordinate projections from the product probability space. Define the Gaussian multiplier process

$$\tilde{\mathbb{G}}_n(g) := \tilde{\mathbb{G}}_n(X_1^n, \xi_1^n)(g) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i(g(X_i) - \mathbb{E}_n[g(X_i)]), \quad g \in \mathcal{G}$$

and for $x_1^n \in S^n$, let $\tilde{W}_n(x_1^n) := \|\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{G}}$ denote the supremum of this process calculated for fixed $X_1^n = x_1^n$. In addition, let

$$\psi_n := \sqrt{\frac{\sigma^2 K_n}{n}} + \left(\frac{b^2 \sigma^2 K_n^3}{n} \right)^{1/4} \quad \text{and} \quad \gamma_n(\delta) := \frac{1}{\delta} \left(\frac{b^2 \sigma^2 K_n^3}{n} \right)^{1/4} + \frac{1}{n}.$$

The following theorem shows that $\tilde{W}_n(X_1^n)$ can be well approximated by the supremum of the Gaussian process B under mild conditions on b , σ , and K_n .

Theorem 3.2. *Consider the setting specified above. Assume that $b^2 K_n \leq n \sigma^2$. Then for any $\delta > 0$, there exists a set $S_{n,0} \in S^n$ such that $P(S_{n,0}) \geq 1 - 3/n$ and for any $x_1^n \in S_{n,0}$ one can construct on an enriched probability space a random variable W^0 such that (i) $W^0 \stackrel{d}{=} \|B\|_{\mathcal{G}}$ and (ii)*

$$P(|\tilde{W}_n(x_1^n) - W^0| > (\psi_n + \delta)) \leq A \gamma_n(\delta)$$

where A is an absolute constant.

Theorems 3.1 and 3.2 combined with anti-concentration inequalities (Theorem 2.1 and Corollary 2.1) can be used to prove validity of Gaussian multiplier bootstrap for approximating distributions of suprema of empirical processes of VC type function classes without weak convergence arguments. This allows us to cover cases where complexity of the function class \mathcal{G} is increasing with n , which is typically the case in nonparametric problems in general and in confidence band construction in particular. Moreover, approximation error can be shown to be polynomially (in n) small under mild conditions. In the next two sections, we will demonstrate how to use these theorems for honest adaptive confidence band construction.

4. ANALYSIS OF CONFIDENCE BANDS UNDER HIGH-LEVEL CONDITIONS

We go back to the analysis of confidence bands. Recall that we consider the following setting. We observe i.i.d. random variables X_1, \dots, X_n with common unknown density $f \in \mathcal{F}$ on \mathbb{R}^d , where \mathcal{F} is a nonempty subset of densities on \mathbb{R}^d . We denote by P_f the probability distribution corresponding to the density f . We assume that the variables X_i are defined as coordinate projections from the product space. We will derive the theory under the following conditions.

4.1. Conditions. Let $\mathcal{X} \subset \mathbb{R}^d$ be a set of interest. Let $\hat{f}_n(\cdot, l)$ be a generic estimator of f with a smoothing parameter $l \in \mathcal{L}_n$. Denote by $\sigma_{n,f}(x, l)$ the standard deviation of $\sqrt{n}\hat{f}_n(x, l)$. We assume that $\sigma_{n,f}(x, l)$ is strictly positive on $\mathcal{V}_n := \mathcal{X} \times \mathcal{L}_n$ for all $f \in \mathcal{F}$. Define the studentized process $Z_{n,f} = \{Z_{n,f}(v) : v = (x, l) \in \mathcal{V}_n\}$ by (1.2). To avoid a measurability problem, we assume that the process $Z_{n,f}$ is separable. Denote $W_{n,f} := \|Z_{n,f}\|_{\mathcal{V}_n}$. Let c_1 and C_1 be some positive constants.

Condition H1 (Gaussian approximation). *For each $f \in \mathcal{F}$, one can construct on an enriched probability space a sequence of random variables $W_{n,f}^0$ such that (i) $W_{n,f}^0 \stackrel{d}{=} \|G_{n,f}\|_{\mathcal{V}_n}$ where $G_{n,f} = \{G_{n,f}(v) : v \in \mathcal{V}_n\}$ is a centered tight Gaussian process with $E[G_{n,f}(v)^2] = 1$ for all $v \in \mathcal{V}_n$ and $E[\|G_{n,f}\|_{\mathcal{V}_n}] < C_1\sqrt{\log n}$ and (ii)*

$$\sup_{f \in \mathcal{F}} P_f(|W_{n,f} - W_{n,f}^0| > \epsilon_{1n}) \leq \delta_{1n},$$

where ϵ_{1n} and δ_{1n} are some sequences of positive numbers bounded from above by $C_1 n^{-c_1}$.

Let $\alpha \in (0, 1)$ be a fixed constant (confidence level). Recall that $c_{n,f}(\alpha)$ is the $(1 - \alpha)$ -quantile of the random variable $\|G_{n,f}\|_{\mathcal{V}_n}$. If $G_{n,f}$ is pivotal, i.e., independent of f , $c_{n,f}(\alpha) = c_n(\alpha)$ can be directly computed, at least numerically. Otherwise, we have to approximate or estimate $c_{n,f}(\alpha)$. Let $\hat{c}_n(\alpha)$ be a generic estimator or approximated value of $c_{n,f}(\alpha)$. The theory in the next section assumes that $\hat{c}_n(\alpha)$ is obtained via Gaussian multiplier bootstrap simulations.

Condition H2 (Estimation error of $\hat{c}_n(\alpha)$). *For some sequences τ_n , ϵ_{2n} , and δ_{2n} of positive numbers bounded from above by $C_1 n^{-c_1}$, we have*

$$(a) \sup_{f \in \mathcal{F}} P_f(\hat{c}_n(\alpha) < c_{n,f}(\alpha + \tau_n) - \epsilon_{2n}) \leq \delta_{2n};$$

$$(b) \sup_{f \in \mathcal{F}} P_f(\hat{c}_n(\alpha) > c_{n,f}(\alpha - \tau_n) + \epsilon_{2n}) \leq \delta_{2n}.$$

Let $\hat{\sigma}_n(x, l)$ be a generic estimator of $\sigma_{n,f}(x, l)$. Without loss of generality, we may assume that $\hat{\sigma}_n(x, l)$ is nonnegative. Condition H3 below states a high-level assumption on the estimation error of $\hat{\sigma}_n(x, l)$. Verifying Condition H3 is rather standard for specific examples.

Condition H3 (Estimation error of $\hat{\sigma}_n(\cdot)$). *For some sequences ϵ_{3n} and δ_{3n} of positive numbers bounded from above by $C_1 n^{-c_1}$,*

$$\sup_{f \in \mathcal{F}} P_f \left(\sup_{v \in \mathcal{V}_n} \left| \frac{\hat{\sigma}_n(v)}{\sigma_{n,f}(v)} - 1 \right| > \epsilon_{3n} \right) \leq \delta_{3n}.$$

We assume that the smoothing parameter $\hat{l}_n := \hat{l}_n(X_1, \dots, X_n)$, which is allowed to depend on the data, is chosen so that the bias can be controlled

sufficiently well. Specifically, for all $l \in \mathcal{L}_n$, define

$$\Delta_{n,f}(l) := \sup_{x \in \mathcal{X}} \frac{\sqrt{n}|f(x) - \mathbb{E}_f[\hat{f}_n(x, l)]|}{\hat{\sigma}_n(x, l)}.$$

We assume that there exists a sequence of random numbers c'_n , which are known or can be calculated via simulations, that control $\Delta_{n,f}(\hat{l}_n)$. The theory in the next section assumes that c'_n is chosen as a multiple of the estimated high quantile of $\|G_{n,f}\|_{\mathcal{V}_n}$.

Condition H4 (Bound on $\Delta_{n,f}(\hat{l}_n)$). *For some sequence δ_{4n} of positive numbers bounded from above by $C_1 n^{-c_1}$,*

$$\sup_{f \in \mathcal{F}} P_f \left(\Delta_{n,f}(\hat{l}_n) > c'_n \right) \leq \delta_{4n}.$$

For the purposes of the analysis of the length of confidence bands, we assume that c'_n in turn can be controlled by $u_n \sqrt{\log n}$ where u_n is a sequence of positive numbers. Typically, u_n is either a bounded or slowly growing sequence.

Condition H5 (Bound on c'_n). *For some sequences δ_{5n} and u_n of positive numbers where δ_{5n} is bounded from above by $C_1 n^{-c_1}$ and u_n is bounded from below by c_1 ,*

$$\sup_{f \in \mathcal{F}} P_f \left(c'_n > u_n \sqrt{\log n} \right) \leq \delta_{5n}.$$

If the function class \mathcal{F} where the true density f belongs to is known, it follows from the theory in the next section that one can find a constant $C(\mathcal{F})$ depending on \mathcal{F} only such that Condition H5 can be satisfied by setting $u_n = C(\mathcal{F})$. In applications, however, \mathcal{F} is usually unknown. In these cases, one can assume that u_n is slowly growing, so that $u_n \geq C(\mathcal{F})$ in sufficiently large samples.

Finally, we assume the following condition on the growth of $\hat{\sigma}_n(\cdot, \hat{l}_n)$.

Condition H6 (Bound on $\hat{\sigma}_n(x, \hat{l}_n)$). *There exists a function $t : \mathcal{F} \rightarrow \mathbb{R}$ and a sequence δ_{6n} of positive numbers bounded from above by $C_1 n^{-c_1}$ such that*

$$\sup_{f \in \mathcal{F}} P_f \left(\sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, \hat{l}_n)^2 > C_1 \left(\frac{\log n}{n} \right)^{-d/(2t(f)+d)} \right) \leq \delta_{6n}.$$

The function $t(f)$ measures the smoothness of the density f . For example, if \mathcal{F} is a suitable subset of a Holder ball, then $t(f)$ is equal to the Holder order of the density f .

4.2. Results.

$$\begin{aligned} \bar{\epsilon}_{n,f} &:= \epsilon_{1n} + \epsilon_{2n} + \epsilon_{3n} c_{n,f}(\alpha), \\ \delta_n &:= \delta_{1n} + \delta_{2n} + \delta_{3n} + \delta_{4n}. \end{aligned}$$

We now state the main results of this section.

Proposition 4.1. *Assume that Conditions H1-H4 are satisfied. Consider the confidence band $C_n = \{C_n(x) : x \in \mathcal{X}\}$ defined by*

$$C_n(x) := \left[\hat{f}_n(x, \hat{l}_n) - s_n(x, \hat{l}_n), \hat{f}_n(x, \hat{l}_n) + s_n(x, \hat{l}_n) \right] \quad (4.1)$$

where

$$s_n(x, \hat{l}_n) := (\hat{c}_n(\alpha) + c'_n) \frac{\hat{\sigma}_n(x, \hat{l}_n)}{n^{1/2}} \quad (4.2)$$

for all $x \in \mathcal{X}$. Then, we have for all $n \geq 1$ and $f \in \mathcal{F}$,

$$\mathbb{P}_f(f \in C_n) \geq (1 - \alpha) - \delta_n - \tau_n - A\bar{\epsilon}_{n,f} \mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}]$$

for some absolute constant A . In particular, since $\mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}] \leq C_1 \sqrt{\log n}$, $\mathbb{P}_f(f \in C_n) \geq 1 - \alpha - Cn^{-c}$ for some constants c and C depending on c_1 and C_1 only.

Comment 4.1. (i) Proposition 4.1 shows that the confidence bands in (4.1) are asymptotically honest with level α for the class \mathcal{F} . Moreover, since the constants c and C in the statement of the proposition depend on c_1 and C_1 only, the coverage probability can be smaller than $1 - \alpha$ only by a polynomially small term Cn^{-c} uniformly over the class \mathcal{F} .

(ii) An advantage of Proposition 4.1 is that it does not require Smirnov-Bickel-Rosenblatt (SBR) condition that is often difficult to obtain. In particular, in the next section we will show that our proposition applies when $\hat{f}_n(\cdot)$ is defined using either convolution or projection kernels under mild conditions, and, as far as projection kernels are concerned, covers estimators based on compactly supported wavelets, Battle-Lemarie wavelets of *any* order as well as other estimators such as those based on orthogonal polynomials and trigonometric functions. SBR condition for compactly supported wavelets was obtained in [4], for Battle-Lemarie wavelets of degree upto 4 in [14], and for Battle-Lemarie wavelets of degree higher than 4 in [10]. To the best of our knowledge, SBR condition for orthogonal polynomials and trigonometric functions has not been obtained in the literature. In addition, SBR condition, being based on extreme value theory, yields only a logarithmic (in n) rate of approximation of coverage probability. In contrast, our proposition gives polynomial rate.

Proposition 4.2. *Assume that Conditions H1-H6 are satisfied. Consider the confidence band $C_n := \{C_n(x) : x \in \mathcal{X}\}$ defined in (4.1). Then*

$$\sup_{f \in \mathcal{F}} \mathbb{P}_f \left(\sup_{x \in \mathcal{X}} \lambda(C_n(x)) > C_1 \bar{c}_n \frac{r_n(f)}{\sqrt{\log n}} \right) \leq \delta_{2n} + \delta_{5n} + \delta_{6n} \quad (4.3)$$

where

$$\bar{c}_n := c_{n,f}(\alpha - \tau_n) + \epsilon_{2n} + u_n \sqrt{\log n}$$

and

$$r_n(f) := \left(\frac{\log n}{n} \right)^{t(f)/(2t(f)+d)}$$

and where λ denotes the Lebesgue measure on \mathbb{R} . In particular, there exists a finite constant L depending on α , c_1 , and C_1 only such that

$$\sup_{f \in \mathcal{F}} P_f \left(\sup_{x \in \mathcal{X}} \lambda(C_n(x)) > Lu_n r_n \right) \leq Cn^{-c} \quad (4.4)$$

for some constants c and C depending on c_1 and C_1 only.

Comment 4.2. Proposition 4.2 shows that the confidence bands in (4.1) adapt to the smoothness $t(f)$ of the density f . When \mathcal{F} is a suitable subset of the Holder ball (so that $t(f)$ equals the Holder order of f) and u_n is bounded, the rate of convergence of the length of confidence bands to zero $u_n r_n$ coincides with the minimax-optimal rate of estimation of f in the uniform metric. No additional inflating terms are required.

5. VERIFYING CONDITIONS H1-H6

In this section, we show that conditions H1-H6 used above hold for estimators $\hat{f}(x, l)$ based on typical convolution and projection kernels under some commonly used assumptions. Recall that the estimators are based on an i.i.d. sample X_1, \dots, X_n of observations with density $f \in \mathcal{F}$ on \mathbb{R}^d . Let ξ_1, \dots, ξ_n be a sequence of $N(0, 1)$ random variables that are independent of $X_1^n := \{X_1, \dots, X_n\}$. Denote $\xi_1^n := \{\xi_1, \dots, \xi_n\}$. The set of random variables ξ_1^n will be used to simulate critical values $\hat{c}(\alpha)$ and c'_n . Throughout the rest of the paper, we assume that both X_1^n and ξ_1^n are defined as coordinate projections from some product probability space.

Let $\{K_l\}_{l \in \mathcal{L}_n}$ be a family of kernels where $K_l : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ and l is a smoothing parameter. We consider estimators of the form

$$\hat{f}_n(x, l) := \mathbb{E}_n[K_l(\cdot, x)] = \frac{1}{n} \sum_{i=1}^n K_l(X_i, x)$$

for all $x \in \mathcal{X}$ and $l \in \mathcal{L}_n$. The variance of $\sqrt{n}\hat{f}_n(x, l)$ is given by

$$\sigma_{n,f}^2(x, l) := \mathbb{E}_f[K_l(X_1 - x)^2] - \mathbb{E}_f[K_l(X_1 - x)]^2.$$

We assume that $\sigma_{n,f}^2(x, l)$ is estimated by

$$\hat{\sigma}_n^2(x, l) := \frac{1}{n} \sum_{i=1}^n (K_l(X_i - x) - \hat{f}_n(x, l))^2 = \frac{1}{n} \sum_{i=1}^n K_l(X_i - x)^2 - \hat{f}_n(x, l)^2$$

for all $x \in \mathcal{X}$ and $l \in \mathcal{L}_n$, which is a sample analogue estimator.

5.1. Conditions. We will verify conditions H1-H6 from the previous section under the following assumptions.

Condition L1. *The function class $\mathcal{K}_{n,f} := \{K_l(\cdot, x)/\sigma_{n,f}(x, l) : (x, l) \in \mathcal{X} \times \mathcal{L}_n\}$ is VC type with constants $a \geq e$ and $v \geq 1$ and constant envelope b_n for all $f \in \mathcal{F}$. In addition, $\mathbb{E}_f[g(X_1)^2] \leq \sigma_n^2 \leq b_n^2$ for all $f \in \mathcal{F}$ and $g \in \mathcal{K}_{n,f}$.*

Condition L2. *There exist strictly positive constants c_2 and C_2 such that (i) for all $f \in \mathcal{F}$ and $l \in \mathcal{L}_n$,*

$$c_2 2^{ld/2} \leq \inf_{x \in \mathcal{X}} \sigma_{n,f}(x, l) \leq \sup_{x \in \mathcal{X}} \sigma_{n,f}(x, l) \leq C_2 2^{ld/2}$$

and (ii) $c_2 \leq \sigma_n^2 \leq C_2$.

Comment 5.1. In this comment, we provide primitive assumptions that suffice for Conditions L1 and L2.

(a) Convolution kernels. Consider a kernel function $K : \mathbb{R} \rightarrow \mathbb{R}$ that (i) has compact support, (ii) is of bounded variation, and (iii) satisfies $\int K(s)ds = 1$. Let $\mathcal{L}_n \subset (0, \infty)$. For $x, y \in \mathbb{R}^d$ and $l \in \mathcal{L}_n$, define

$$K_l(y, x) := 2^{ld} \prod_{1 \leq m \leq d} K\left(2^l(y_m - x_m)\right).$$

Here 2^{-l} can be interpreted as a bandwidth parameter. Suppose that \mathcal{L}_n is contained in the interval $[l_{\min,n}, l_{\max,n}]$ where $l_{\min,n} \rightarrow \infty$ as $n \rightarrow \infty$. In addition, suppose that uniformly over $f \in \mathcal{F}$,

$$f(x) \geq c \text{ for all } x \in \mathcal{X}^c \text{ and } f(x) \leq C \text{ for all } x \in \mathbb{R}^d \quad (5.1)$$

where \mathcal{X}^c is the c -enlargement of \mathcal{X} .

Then $\|K_l(\cdot, x)\|_{\mathbb{R}^d} \leq C 2^{ld}$ for all $x \in \mathbb{R}^d$ and $l \in \mathcal{L}_n$ because K has compact support and is of bounded variation. Further, there exists n_0 such that uniformly over all $f \in \mathcal{F}$, $x \in \mathcal{X}$, and $l \in \mathcal{L}_n$,

$$c 2^{ld} \leq \sigma_{n,f}(x, l)^2 \leq C 2^{ld} \quad (5.2)$$

for all $n \geq n_0$. This implies the first part of condition L2. Since (5.1) gives $|\mathbb{E}_f[K_l(X_1, x)]| \leq C$ uniformly over all $f \in \mathcal{F}$, $l \in \mathcal{L}_n$, and $x \in \mathbb{R}^d$, (5.2) also implies the second part of condition L2. Since the product of VC type classes is VC type, it is also easy to check that condition L1 holds for all $n \geq n_0$ with some a and v independent of n , $b_n \leq C 2^{l_{\max,n} d/2}$. See [9] for a more general class of kernels that satisfy conditions L1 and L2.

(ii) Projection kernels: compactly supported wavelets. Consider a father wavelet ϕ , i.e. a function ϕ such that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$, the spaces $V_j = \{\sum_k c_k \phi(2^j x - k) : \sum_k c_k^2 < \infty\}$, $j = 0, 1, 2, \dots$, are nested, and $\cup_{j \geq 0} V_j$ is dense in $L^2(\mathbb{R})$. Suppose that ϕ is compactly supported, bounded, and of bounded p -variation for some $p \geq 1$. For example, all Daubechies' wavelets have these properties (see [12], p. 1613). Suppose also that there exists a constant $c > 0$ such that for all $x \in \mathbb{R}^d$,

$$\sum_{k \in \mathbb{Z}^d} \left(\prod_{1 \leq m \leq d} \phi(x_m - k_m) \right)^2 \geq c. \quad (5.3)$$

Let $\mathcal{L}_n \subset \mathbb{N}$. For $x, y \in \mathbb{R}^d$ and $l \in \mathcal{L}_n$, define

$$K_l(y, x) := 2^{ld} \sum_{k \in \mathbb{Z}^d} \prod_{1 \leq m \leq d} \phi(2^l y_m - k_m) \prod_{1 \leq m \leq d} \phi(2^l x_m - k_m) \quad (5.4)$$

Since ϕ is compactly supported and bounded, $|2^{-ld}K_l(y, x)| \leq C$ uniformly over all $x, y \in \mathbb{R}^d$ and $l \in \mathcal{L}_n$. Therefore, it follows from Lemma 2 in [12] that the function class $\bar{\mathcal{K}}^d := \{2^{-ld}K_l(\cdot, x) : l \in \mathbb{N}, x \in \mathbb{R}^d\}$ is VC type with constant envelope when $d = 1$. When $d \geq 2$, $\bar{\mathcal{K}}^d$ can be represented as a product of classes $\bar{\mathcal{K}}^1$ corresponding to different coordinates of \mathbb{R}^d , and so it is also VC type with constant envelope.

Suppose that \mathcal{L}_n is contained in the interval $[l_{\min,n}, l_{\max,n}]$ where $l_{\min,n} \rightarrow \infty$ as $n \rightarrow \infty$. In addition, suppose that uniformly over all $f \in \mathcal{F}$, (5.1) holds. Since ϕ has a compact support, $K_l(y, x) = 0$ if $|2^l y - 2^l x| \geq c$ for some constant $c > 0$. Therefore, $|\mathbb{E}_f[K_l(X_i, x)]| \leq C$ because $2^{-ld}|K_l(X_i, x)|$ is bounded. Similarly, $\mathbb{E}_f[K_l(X_i, x)^2] \leq 2^{ld}C$. Further, there exists n_0 such that for all $n \geq n_0$ uniformly over all $f \in \mathcal{F}$, $x \in \mathcal{X}$, and $l \in \mathcal{L}_n$,

$$\begin{aligned} \mathbb{E}_f[K_l(X_i, x)^2] &= \int_{\mathbb{R}^d} K_l(y, x)^2 f(y) dy = \int_{y:|y-x|\leq 2^{-l}c} K_l(y, x)^2 f(y) dy \\ &\geq c \int_{y:|y-x|\leq 2^{-l}c} K_l(y, x)^2 dy = c \int_{\mathbb{R}^d} K_l(y, x)^2 dy \\ &= 2^{ld}c \sum_{k \in \mathbb{Z}^d} \left(\prod_{1 \leq m \leq d} \phi(2^l x_m - k_m) \right)^2 \geq 2^{ld}c \end{aligned}$$

where the last inequality follows from assumption (5.3). Therefore, for all $n \geq n_0$,

$$2^{ld}c \leq \sigma_{n,f}(x, l)^2 \leq 2^{ld}C,$$

which implies condition L2 as in the case of convolution kernels. Since the product of VC type classes is VC type, we conclude that condition L1 also holds for all $n \geq n_0$ with some constants a and v independent of n , $b_n = C2^{l_{\max,n}d/2}$.

(iii) Projection kernels: Battle-Lemarie wavelets. Consider a Battle-Lemarie farther wavelet ϕ of order $r \geq 1$. Suppose that (5.3) holds for some $c > 0$. Let $\mathcal{L}_n \subset \mathbb{N}$. For $x, y \in \mathbb{R}^d$ and $l \in \mathcal{L}_n$, define $K_l(y, x)$ by (5.4). It follows from Lemma 1 in [15] that $|2^{-ld}K_l(y, x)| \leq C$ uniformly over all $x, y \in \mathbb{R}^d$ and $l \in \mathcal{L}_n$. Therefore, it follows from Lemma 2 in [15] that the function class $\bar{\mathcal{K}}^d := \{2^{-ld}K_l(\cdot, x) : l \in \mathbb{N}, x \in \mathbb{R}^d\}$ is VC type with constant envelope when $d = 1$. When $d \geq 2$, $\bar{\mathcal{K}}^d$ can be represented as a product of classes $\bar{\mathcal{K}}^1$ corresponding to different coordinates of \mathbb{R}^d , and so it is also VC type with constant envelope.

Suppose that \mathcal{L}_n is contained in the interval $[l_{\min,n}, l_{\max,n}]$ where $l_{\min,n} \rightarrow \infty$ as $n \rightarrow \infty$. In addition, suppose that uniformly over all $f \in \mathcal{F}$, (5.1) holds. Since ϕ is a Battle-Lemarie wavelet, $2^{-ld}|K_l(y, x)| \leq \exp(-c|2^l y - 2^l x|)$ for some constant $c > 0$. Therefore, $|\mathbb{E}_f[K_l(X_i, x)]| \leq C$ because $2^{-ld}|K_l(X_i, x)|$ is bounded. Similarly, $\mathbb{E}_f[K_l(X_i, x)^2] \leq 2^{ld}C$. Further, there exists n_0 such that for all $n \geq n_0$ uniformly over all $f \in \mathcal{F}$, $x \in \mathcal{X}$, and

$l \in \mathcal{L}_n$,

$$\begin{aligned} \mathbb{E}_f[K_l(X_i, x)^2] &= \int_{\mathbb{R}^d} K_l(y, x)^2 f(y) dy \geq \int_{y:|y-x|\leq c} K_l(y, x)^2 f(y) dy \\ &\geq c \int_{y:|y-x|\leq c} K_l(y, x)^2 dy = c \int_{\mathbb{R}^d} K_l(y, x)^2 dy - C \\ &= 2^{ld} c \sum_{k \in \mathbb{Z}^d} \left(\prod_{1 \leq m \leq d} \phi(2^l x_m - k_m) \right)^2 - C \geq 2^{ld} c \end{aligned}$$

where the last inequality follows from assumption (5.3) and the fact that all $l \in \mathcal{L}_n$ are sufficiently large since $n \geq n_0$ and $l_{\min,n} \rightarrow \infty$. Therefore, for all $n \geq n_0$,

$$2^{ld} c \leq \sigma_{n,f}(x, l)^2 \leq 2^{ld} C,$$

which implies condition L2 as in the case of convolution kernels. Since the product of VC type classes is VC type, we conclude that condition 1 also holds for all $n \geq n_0$ with some constants a and v independent of n , $b_n = C 2^{l_{\max,n} d/2}$, and $c \leq \sigma_n^2 \leq C$.

(iv) Projection kernels: other bases. Suppose that \mathcal{X} equals the whole support of f for all $f \in \mathcal{F}$.² Let $\{\varphi_j : j = 1, \dots, \infty\}$ be an orthonormal basis of $L_2(\mathcal{X})$, the space of square integrable functions on \mathcal{X} . Let $\mathcal{L}_n \subset (0, \infty)$ be such that $2^{ld} \in \mathbb{N}$ for all $l \in \mathcal{L}_n$. For $x, y \in \mathcal{X}$ and $l \in \mathcal{L}_n$, define

$$K_l(y, x) := \sum_{j=1}^{2^{ld}} \varphi_j(y) \varphi_j(x).$$

Here, 2^{ld} equals the number of series (basis) terms used in the estimation. Suppose that \mathcal{L}_n is contained in the interval $[l_{\min,n}, l_{\max,n}]$ where $l_{\min,n} \rightarrow \infty$ as $n \rightarrow \infty$. For all $x \in \mathcal{X}$, let $\Phi_l(x)$ be a $2^{ld} \times 2^{ld}$ matrix with (j, k) -th component equal to $\varphi_j(x) \varphi_k(x)$. Suppose that all eigenvalues of $\mathbb{E}_f[\Phi_l(X)]$ are bounded from above by C and from below by c uniformly over all $l \in \mathcal{L}_n$ and $f \in \mathcal{F}$. In addition, assume that $|\mathbb{E}_f[K_l(X_1, x)]| \leq C$ uniformly over all $l \in \mathcal{L}_n$ and $f \in \mathcal{F}$ and $c 2^{ld} \leq \sum_{j=1}^{2^{ld}} \varphi_j(x)^2 \leq C 2^{ld}$ uniformly over all $l \in \mathcal{L}_n$. Then

$$\begin{aligned} \mathbb{E}_f[(K_l(X_1, x))^2] &\leq C \sum_{j=1}^{2^{ld}} \varphi_j(x)^2 \leq C 2^{ld}, \\ \mathbb{E}_f[(K_l(X_1, x))^2] &\geq c \sum_{j=1}^{2^{ld}} \varphi_j(x)^2 \leq c 2^{ld}, \end{aligned}$$

²The case when \mathcal{X} is a proper subset of the support of f can be handled similarly but requires a more technically involved argument.

and so the first part of Condition L2 holds. Further, for all $x, y \in \mathcal{X}$ and $l \in \mathcal{L}_n$,

$$K_l(y, x) \leq \sqrt{\sum_{j=1}^{2^{ld}} \varphi_j(y)^2 \sum_{j=1}^{2^{ld}} \varphi_j(x)^2} \leq C 2^{ld},$$

and so the second part of Condition L2 follows as well.

Further, assume that \mathcal{X} is compact, and that there exists L_n satisfying $\log n \leq C \log n$ such that $\sum_{j=1}^{2^{ld}} (\varphi_j(x) - \varphi_j(y))^2 \leq L_n^2 \sum_{j=1}^d (x_j - y_j)^2$. Then Condition L1 holds with constants $a = CL_n 2^{l_{\max,n} d/2}$ and some v independent of n , $b_n = C 2^{l_{\max,n} d/2}$, and $c \leq \sigma_n^2 \leq C$.

Following [14], we will impose the following condition restricting the function class \mathcal{F} .

Condition L3. *There exist $n_0 \in \mathbb{N}$ and strictly positive constants c_3, C_3, c_4 , and C_4 such that for all $n \geq n_0$, $f \in \mathcal{F}$ and $l \in \mathcal{L}_n$, there is some $t \in [c_3, C_3]$ such that*

$$c_4 2^{-lt} \leq \sup_{x \in \mathcal{X}} |\mathbf{E}_f[\hat{f}_n(x, l)] - f(x)| \leq C_4 2^{-lt}. \quad (5.5)$$

Comment 5.2. Let $r \in \mathbb{N}$. Assume that $d = 1$. In addition, assume that the estimator $\hat{f}_n(\cdot)$ is constructed either from convolution or wavelet projection kernel as described in comment 5.1. Further, for convolution kernels, assume that $\int K(s)s^l ds = 0$ for $l = 1, \dots, r-1$ and $\int K(s)|s|^r ds$ is finite. For compactly supported wavelet projection kernels, assume that either ϕ is $(r-1)$ -regular or corresponding mother wavelet ψ satisfies $\int \psi(s)s^l ds = 0$ for every $0 \leq l \leq r-1$. For Battle-Lemarie wavelet projection kernels, assume that wavelet of order r is used. Then it is well-known that the upper bound in Condition 3 holds if $f \in \mathcal{C}^t$ for any $t < r$ where \mathcal{C}^t is the Holder-Zygmund space. As far as the lower bound is concerned, [14] showed that it holds for "generically" in Holder spaces with smoothness $t < r$. See the original paper for more detailed explanation of this result.

5.2. Results. Let $K_n := A v(\log n \vee \log(ab_n))$ for some sufficiently large but absolute constant A . Let $G_{n,f} := \{G_{n,f}(v) : v \in \mathcal{V}_n\}$ be a zero mean tight Gaussian process with the same covariance structure as that of $Z_{n,f}$. It is well known that under Condition L1 such a process exists.

For all $x \in \mathcal{X}$ and $l \in \mathcal{L}_n$, define Gaussian multiplier process:

$$\hat{\mathbb{G}}_n(x, l) := \hat{\mathbb{G}}_n(X_1^n, \xi_1^n)(x, l) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \frac{K_l(X_i - x) - \hat{f}_n(x, l)}{\hat{\sigma}_n(x, l)}.$$

We assume that the critical value $\hat{c}_n(\alpha)$ is simulated as conditional $(1 - \alpha)$ -quantile of $\|\hat{\mathbb{G}}_n\|_{\mathcal{V}_n}$ given X_1^n . Let c_5 and C_5 be some strictly positive constants. The following theorem verifies Conditions H1-H3.

Proposition 5.1. *Assume that conditions L1 and L2 hold. Assume that $b_n^2 K_n^4/n \leq C_5 n^{-c_5}$. Then conditions H1-H3 hold with constants c_1 and C_1*

depending on c_2 , C_2 , c_5 , and C_5 only and with the process $G_{n,f}$ and the critical value $\hat{c}_n(\alpha)$ defined above. Moreover, condition H2 holds uniformly over all $\alpha \in (0, 1)$.

Next, we verify conditions H4-H6 assuming that the smoothing parameter \hat{l}_n is chosen according to a version of the Lepski's method. Specifically, let γ_n be a sequence of strictly positive numbers converging to zero. Let $c_{n,f}(\gamma_n)$ be the $(1 - \gamma_n)$ -quantile of the random variable $\|G_{n,f}\|_{\mathcal{V}_n}$, and let $\hat{c}_n(\gamma_n)$ be an estimator of $c_{n,f}$. We assume that $\hat{c}_n(\gamma_n)$ satisfies condition H2 with α replaced by γ_n , which holds under conditions of Proposition 5.1. For all $l \in \mathcal{L}_n$, let $\mathcal{L}_{n,l} := \{l' \in \mathcal{L}_n : l' > l\}$. For some constant $q > 1$, which is independent of n , define a "Lepski-type" estimator

$$\hat{l}_n := \inf \left\{ l \in \mathcal{L}_n : \sup_{l' \in \mathcal{L}_{n,l}} \sup_{x \in \mathcal{X}} \frac{\sqrt{n} |\hat{f}_n(x, l) - \hat{f}_n(x, l')|}{\hat{\sigma}_n(x, l) + \hat{\sigma}_n(x, l')} \leq q \hat{c}_n(\gamma_n) \right\} \quad (5.6)$$

Previous literature on the Lepski's estimator used Talagrand's inequality combined with some bounds on expectations of suprema of certain empirical processes (obtained via either entropy methods or Rademacher averages) to choose the threshold level for the estimator (the right hand side of the inequality in (5.6)); see [13] and [15]. Because of the one-sided nature of the aforementioned inequalities, however, it was argued that the resulting threshold turned out to be too high leading to limited applicability of the estimator in small and moderate samples. In contrast, an advantage of our construction is that we use $q \hat{c}_n(\gamma_n)$ as a threshold level, which is essentially the minimal possible value of the threshold that suffices for good properties of the estimator. The analysis of theoretical consequences of our construction beyond the fact that it is sufficient for our results is out of the scope of this paper.

Let u_n be a sequence of positive numbers such that u_n is sufficiently large for large n . See the theorem below for the exact requirements on u_n . We set $c'_n := u_n \hat{c}_n(\gamma_n)$. The following theorem verifies Condition H4-H6.

Proposition 5.2. *Assume that conditions L1, L2, and L3 hold. In addition, assume that $b_n^2 K_n^4/n \leq C_5 n^{-c_5}$, $\gamma \leq C_5 n^{-c_5}$, $|\log \gamma| \leq C_5 \log n$, and $u \geq c_5 \log n$. Finally, assume that \mathcal{L}_n is closed and that there exists $s > 0$ such that for any $l \in \mathcal{L}_n$, there either exists $l' \in \mathcal{L}_n$ satisfying $l' \in (l-s, l)$ or there is no $l' \in \mathcal{L}_n$ such that $l' < l$. Then conditions H4-H6 hold with \hat{l}_n and c'_n defined above and constants c_1 and C_1 depending on $\{(c_j, C_j) : 2 \leq j \leq 5\}$ only.*

APPENDIX A. TECHNICAL TOOLS

Theorem A.1. *Let X and Y be zero-mean Gaussian p -vectors with covariances Σ^X and Σ^Y correspondingly. Then for any $g \in C^2(\mathbb{R})$,*

$$\left| \mathbb{E} \left[g \left(\max_{1 \leq j \leq p} X_j \right) \right] - \mathbb{E} \left[g \left(\max_{1 \leq j \leq p} Y_j \right) \right] \right| \leq \|g''\|_\infty \Delta / 2 + 2 \|g'\|_\infty \sqrt{2 \Delta \log p}$$

where $\Delta = \max_{1 \leq j, k \leq p} |\Sigma_{jk}^X - \Sigma_{jk}^Y|$.

Proof. See Theorem 1 in [7]. \square

Theorem A.2. Let μ and ν be Borel probability measures on \mathbb{R} . Let $\varepsilon > 0$ and $\delta > 0$. Suppose that $\mu(A) \leq \nu(A^\delta) + \varepsilon$ for every Borel subset A of \mathbb{R} . Let V be a random variable with distribution μ . Then there is a random variable W with distribution ν such that $P(|V - W| > \delta) \leq \varepsilon$.

Proof. See Lemma 4.1 in [6]. \square

Theorem A.3. Let $\beta > 0$ and $\delta > 1/\beta$. For every Borel subset B of \mathbb{R} , there is a smooth function $g : \mathbb{R} \rightarrow \mathbb{R}$ and absolute constant $A > 0$ such that $\|g'\|_\infty \leq \delta^{-1}$, $\|g''\|_\infty \leq A\beta\delta^{-1}$, and for all $t \in \mathbb{R}$

$$(1 - \varepsilon)1_B(t) \leq g(t) \leq \varepsilon + (1 - \varepsilon)1_{B^{3\delta}}(t)$$

where $\varepsilon = \varepsilon_{\beta, \delta}$ is given by

$$\varepsilon = \sqrt{e^{-\alpha}(1 + \alpha)} < 1, \quad \alpha = \beta^2\delta^2 - 1.$$

Proof. See Lemma 4.2 in [6]. \square

Theorem A.4. Let ξ_1, \dots, ξ_n be i.i.d. random variables taking values in a measurable space (S, \mathcal{S}) . Suppose that \mathcal{G} is a nonempty, pointwise measurable class of functions on S uniformly bounded by a constant b such that there exist constants $a \geq e$ and $v > 1$ with $\sup_Q N(\mathcal{G}, L_2(Q), b\epsilon) \leq (a/\epsilon)^v$ for all $0 < \epsilon \leq 1$. Let σ^2 be a constant such that $\sup_{g \in \mathcal{G}} \text{var}(g) \leq \sigma^2 \leq b^2$. If $b^2 v \log(ab/\sigma) \leq n\sigma^2$, then for all $t \leq n\sigma^2/b^2$,

$$P \left[\sup_{g \in \mathcal{G}} \left| \sum_{i=1}^n \{g(\xi_i) - E[g(\xi_1)]\} \right| > A \sqrt{n\sigma^2 \left\{ t \vee \left(v \log \frac{ab}{\sigma} \right) \right\}} \right] \leq e^{-t},$$

where $A > 0$ is an absolute constant.

Proof. This version of Talagrand's inequality follows from Theorem 3 in [21] combined with a bound on expected values of suprema of empirical processes derived in [9]. \square

Theorem A.5. Let $Y := \{Y(t) : t \in T\}$ be a zero mean separable Gaussian process such that $E[Y(t)^2] = 1$ for all $t \in T$. Let $c(\alpha)$ denote the $(1 - \alpha)$ -quantile of $\|Y\|_T$. Assume that $E[\|Y\|_T] < \infty$. Then $c(\alpha) \leq E[\|Y\|_T] + \sqrt{2|\log \alpha|}$ and $c(\alpha) \leq M(\|Y\|_T) + \sqrt{2|\log \alpha|}$ for all $\alpha \in (0, 1)$ where $M(\|Y\|_T)$ is the median of $\|Y\|_T$.

Proof. Pick any $\alpha \in (0, 1)$. Since $E[Y(t)^2] = 1$ for all $t \in T$, Borell's inequality (see Theorem A.2.1 in [27]) gives for all $r > 0$,

$$P \{ \|Y\|_T \geq E[\|Y\|_T] + r \} \leq e^{-r^2/2}.$$

Setting $r = \sqrt{2|\log \alpha|}$ gives

$$P \left\{ \|Y\|_T \geq E[\|Y\|_T] + \sqrt{2|\log \alpha|} \right\} \leq \alpha.$$

This implies that $c(\alpha) \leq \mathbb{E}[\|Y\|_T] + \sqrt{2|\log \alpha|}$. The result with $M(\|Y\|_T)$ follows similarly because Borell's inequality also applies with $M(\|Y\|_T)$ replacing $\mathbb{E}[\|Y\|_T]$. \square

APPENDIX B. PROOFS FOR SECTION 2

Proof of Theorem 2.1. The fact that $a(X) < \infty$ follows from Landau-Shepp-Fernique theorem (see, for example, Lemma 2.2.5 in [8]). Since $\sup_{t \in T} X_t \geq X_{t_0}$ for any fixed $t_0 \in T$, $a(X) \geq \mathbb{E}[X_{t_0}] = 0$. We now prove (2.3).

Since the Gaussian process $X = (X_t)_{t \in T}$ is separable, there exists a sequence of finite subsets $T_n \subset T$ such that $Z_n := \max_{t \in T_n} X_t \rightarrow \sup_{t \in T} X_t =: Z$ a.s. as $n \rightarrow \infty$. Fix any $x \in \mathbb{R}$. Since $|Z_n - x| \rightarrow |Z - x|$ a.s. and a.s. convergence implies weak convergence, there exists an at most countable subset \mathcal{N}_x of \mathbb{R} such that for all $\epsilon \in \mathbb{R} \setminus \mathcal{N}_x$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|Z_n - x| \leq \epsilon) = \mathbb{P}(|Z - x| \leq \epsilon).$$

But by Theorem 3 in [7],

$$\mathbb{P}(|Z_n - x| \leq \epsilon) \leq 4\epsilon(\mathbb{E}[\max_{t \in T_n} X_t] + 1) \leq 4\epsilon(a(X) + 1)$$

for all $x \in \mathbb{R}$ and $\epsilon \geq 0$. Therefore,

$$\mathbb{P}(|Z - x| \leq \epsilon) \leq A\epsilon(a(X) \vee 1) \tag{B.1}$$

for all $x \in \mathbb{R}$ and $\epsilon \in \mathbb{R} \setminus \mathcal{N}_x$. By right continuity of $\mathbb{P}(|Z - x| \leq \cdot)$, it follows that (B.1) holds for all $\epsilon \geq 0$, and so (2.3) holds as well. This completes the proof of the theorem. \square

Proof of Corollary 2.1. The proof is analogous to that of Theorem 2.1 and therefore is omitted. \square

APPENDIX C. PROOFS FOR SECTION 3

Proof of Theorem 3.1. The proof consists of applying Theorem 2.1 in [6]. Standard calculations show that for any $\varepsilon \in (0, 1)$,

$$J(\varepsilon) := \int_0^\varepsilon \sup_Q \sqrt{1 + \log N(\mathcal{G}, L_2(Q), b\tau)} d\tau \leq C\varepsilon \sqrt{\log(a/\varepsilon)^v}.$$

For some sufficiently large but absolute constant C , let $\kappa := C(b^3 K_n/n + b\sigma^2)^{1/3}$. Lemma 2.2 in [6] implies that $\kappa^3 \geq \mathbb{E}[\sup_{g \in \mathcal{G}} n^{-1} \sum_{i=1}^n |g(X_i)|^3]$. Let $\varepsilon = \sigma/(bn^{1/2})$. Then $H_n(\varepsilon) := \log(\sup_Q N(\mathcal{G}, L_2(Q), b\varepsilon) \vee n) \leq K_n$ and $J(\varepsilon) \leq C\sigma K_n^{1/2}/(bn^{1/2})$. Note that selecting C in the definition of κ sufficiently large yields $b/\kappa < \gamma^{-1/3} n^{1/3} H_n^{-1/3}$. Therefore, with this choice of parameters, the claim of the theorem follows by applying Theorem 2.1 in [6] (with some intermediate calculations taken from Lemma 2.2 in [6]) using the facts that $K_n \geq 1$, $b \geq \sigma$, and $\gamma < 1$. \square

Proof of Theorem 3.2. Define $\mathcal{G} \cdot \mathcal{G} = \{g \cdot \tilde{g} : g, \tilde{g} \in \mathcal{G}\}$ and $(\mathcal{G} - \mathcal{G})^2 = \{(g - \tilde{g})^2 : g, \tilde{g} \in \mathcal{G}\}$. It is easy to see that $\mathcal{G} \cdot \mathcal{G}$ is VC type with constants $2a$ and $2v$ and constant envelope b^2 and $(\mathcal{G} - \mathcal{G})^2$ is VC type with constants $4a$ and $4v$ and constant envelope $4b^2$. In addition, $E[g^2] \leq b^2\sigma^2$ for all $g \in \mathcal{G} \cdot \mathcal{G}$ and $E[g^2] \leq 16b^2\sigma^2$ for all $g \in (\mathcal{G} - \mathcal{G})^2$. Together with condition $b^2K_n \leq n\sigma^2$, which is assumed, this justifies an application of Talagrand's inequality (Theorem A.4) with $t = \log n$, which gives

$$P \left(\sup_{g \in \mathcal{G}} |\mathbb{E}_n[g(X_i)] - E[g(X_1)]| \leq \sqrt{\frac{\sigma^2 K_n}{n}} \right) \geq 1 - \frac{1}{n}, \quad (\text{C.1})$$

$$P \left(\sup_{g \in \mathcal{G} \cdot \mathcal{G}} |\mathbb{E}_n[g(X_i)] - E[g(X_1)]| \leq \sqrt{\frac{b^2\sigma^2 K_n}{n}} \right) \geq 1 - \frac{1}{n}, \quad (\text{C.2})$$

$$P \left(\sup_{g \in (\mathcal{G} - \mathcal{G})^2} |\mathbb{E}_n[g(X_i)] - E[g(X_1)]| \leq \sqrt{\frac{b^2\sigma^2 K_n}{n}} \right) \geq 1 - \frac{1}{n}. \quad (\text{C.3})$$

Let $S_{n,0} \subset S^n$ be the intersection of events in (C.1)-(C.3). Then $P(S_{n,0}) \geq 1 - 3/n$.

Fix any $x_1^n \in S_{n,0}$. Let $\tau = \sigma/(bn^{1/2})$, and let $\{g_1, \dots, g_N\} \subset \mathcal{G}$ be a subset of elements of \mathcal{G} such that for any $g \in \mathcal{G}$ there exists $j = j(g) \in \{1, \dots, N\}$ such that $E[(g(X_i) - g_j(X_i))^2] \leq b^2\tau^2$. We may and will assume that $N \leq (a/\tau)^v$. Define

$$\begin{aligned} W(x_1^n)(\tau) &:= \max_{1 \leq j \leq N} |\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)(g_j)|, \\ W^0(\tau) &:= \max_{1 \leq j \leq N} |B(g_j)|. \end{aligned}$$

In addition, define $\tilde{W}^0 := \|B\|_{\mathcal{G}}$ and

$$\mathcal{G}(\tau) := \{g - \tilde{g} : g, \tilde{g} \in \mathcal{G}, E[(g(X_i) - \tilde{g}(X_i))^2] \leq b^2\tau^2\}.$$

Clearly, we have $|\tilde{W}_n(x_1^n) - W(x_1^n)(\tau)| \leq \|\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{G}(\tau)}$ and $|\tilde{W}^0 - W^0(\tau)| \leq \|B\|_{\mathcal{G}(\tau)}$. The rest of the proof consists of 3 steps. Steps 1 and 2 provide bounds on $\|\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{G}(\tau)}$ and $\|B\|_{\mathcal{G}(\tau)}$, respectively. Step 3 gives a coupling inequality and finishes the proof using a method for comparing $W(x_1^n)(\tau)$ and $W^0(\tau)$.

Step 1 (Bound on $\|\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{G}(\tau)}$). Here we show that with probability at least $1 - 2/n$,

$$\|\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{G}(\tau)} \leq \sqrt{\frac{\sigma^2 K_n}{n}} + \left(\frac{b^2\sigma^2 K_n^3}{n} \right)^{1/4} = \psi_n.$$

Note that

$$\sup_{g \in \mathcal{G}(\tau)} |\mathbb{E}_n[g(x_i)^2] - \mathbb{E}_n[g(x_i)]^2| \leq \sup_{g \in \mathcal{G}(\tau)} \mathbb{E}_n[g(x_i)^2] := D(\tau).$$

Then $D(\tau) \leq p_1 + p_2 \leq \sigma^2/n + \sqrt{b^2\sigma^2K_n/n}$ because

$$\begin{aligned} p_1 &:= \sup_{g \in \mathcal{G}(\tau)} \mathbb{E}[g(X_1)^2] \leq b^2\tau^2 = \sigma^2/n, \\ p_2 &:= \sup_{g \in \mathcal{G}(\tau)} |\mathbb{E}_n[g(x_i)^2] - \mathbb{E}[g(X_1)^2]| \leq \sup_{g \in (\mathcal{G}-\mathcal{G})^2} |\mathbb{E}_n[g(x_i)] - \mathbb{E}[g(X_1)]| \\ &\leq \sqrt{b^2\sigma^2K_n/n}. \end{aligned}$$

By Borell's inequality (see Theorem A.2.1 in [27]), with probability at least $1 - 2/n$,

$$\|\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{G}(\tau)} \leq \mathbb{E} \left[\|\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{G}(\tau)} \right] + \sqrt{2D(\tau) \log n}.$$

Further, $\mathbb{E}[\|\tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{G}(\tau)}] \leq C(r_1 + r_2)$ where

$$\begin{aligned} r_1 &:= \mathbb{E} \left[\sup_{g \in \mathcal{G}(\tau)} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i g(x_i) \right| \right], \\ r_2 &:= \sup_{g \in \mathcal{G}(\tau)} |\mathbb{E}_n[g(x_i)]|. \end{aligned}$$

To bound r_1 , let $\varphi = \sigma/(bn^{1/2}) + (\sigma^2K_n/(b^2n))^{1/4}$. Note that $\sqrt{D(\tau)}/b \leq \varphi$ and $\varphi \leq 1 + (K_n/n)^{1/4} \leq 2 < a$. So, by Corollary 2.2.8 in [27],

$$\begin{aligned} r_1 &\leq Cb \int_0^\varphi \sqrt{\sup_Q \log N(\mathcal{G}, L_2(Q), b\varepsilon)} d\varepsilon \leq Cb\varphi \sqrt{\log(a/\varphi)^v} \\ &\leq \sqrt{K_n} \left(\frac{\sigma}{\sqrt{n}} + \left(\frac{b^2\sigma^2K_n}{n} \right)^{1/4} \right). \end{aligned}$$

To bound r_2 , we have

$$r_2 \leq 2 \sup_{g \in \mathcal{G}} |\mathbb{E}_n[g(x_i)] - \mathbb{E}[g(X_1)]| + \sup_{g \in \mathcal{G}(\tau)} |\mathbb{E}[g(X_1)]| \leq 3\sqrt{\sigma^2K_n/n}.$$

Combining these inequalities and increasing constant A in the definition of K_n gives the claim of step 1.

Step 2 (Bound on $\|B\|_{\mathcal{G}(\tau)}$). We show that with probability at least $1 - 2/n$,

$$\|B\|_{\mathcal{G}(\tau)} \leq \sqrt{\frac{\sigma^2K_n}{n}} \leq \psi_n.$$

By Borell's inequality, with probability at least $1 - 2/n$,

$$\|B\|_{\mathcal{G}(\tau)} \leq \mathbb{E}[\|B\|_{\mathcal{G}(\tau)}] + b\tau\sqrt{2\log n}.$$

By Corollary 2.2.8 in [27],

$$\mathbb{E}[\|B\|_{\mathcal{G}(\tau)}] \leq Cb \int_0^\tau \sqrt{\sup_Q \log N(\mathcal{G}, L_2(Q), b\varepsilon)} d\varepsilon \leq Cb\tau \sqrt{\log a/\tau}.$$

Substituting $\tau = \sigma/(bn^{1/2})$ into these inequalities gives the claim of step 2.

Step 3 (Coupling Inequality). This is the main step of the proof. Let $\delta > 0$ and $\beta = 2\sqrt{\log n}/\delta$. Then

$$\varepsilon := \sqrt{e^{1-\beta^2\delta^2}\beta^2\delta^2} \leq C/n.$$

Take any Borel subset B of \mathbb{R} and apply Theorem A.3 to define a function f corresponding to the set B^{ψ_n} , ψ_n -enlargement of the set B , with chosen β and δ . We have for all $t \in \mathbb{R}$,

$$(1 - \varepsilon)1_{B^{\psi_n}}(t) \leq f(t) \leq \varepsilon + (1 - \varepsilon)1_{B^{\psi_n+3\delta}}(t).$$

Further,

$$\Delta := \sup_{g_1, g_2 \in \mathcal{G}} |\Delta_{g_1, g_2}| \leq C \sqrt{\frac{b^2 \sigma^2 K_n}{n}}$$

where

$$\begin{aligned} \Delta_{g_1, g_2} &:= (\mathbb{E}_n[g_1(x_i)g_2(x_i)] - \mathbb{E}_n[g_1(x_i)]\mathbb{E}_n[g_2(x_i)]) \\ &\quad - (\mathbb{E}[g_1(X_1)g_2(X_1)] - \mathbb{E}[g_1(X_1)]\mathbb{E}[g_2(X_1)]). \end{aligned}$$

So, applying Theorem A.1 to $W(x_1^n)(\tau)$ and $W^0(\tau)$ with chosen f gives

$$|\mathbb{E}[f(W(x_1^n)(\tau))] - \mathbb{E}[f(W^0(\tau))]| \leq \frac{C}{\delta^2} \sqrt{\frac{b^2 \sigma^2 K_n \log n}{n}} + \frac{C}{\delta} \left(\frac{b^2 \sigma^2 K_n^3}{n} \right)^{1/4}.$$

We will assume that $b^2 \sigma^2 K_n^3 / (n \delta^4) \leq 1$ (otherwise, the bound claimed in the statement of the theorem is trivial). Then

$$|\mathbb{E}[f(W(x_1^n)(\tau))] - \mathbb{E}[f(W^0(\tau))]| \leq \frac{C}{\delta} \left(\frac{b^2 \sigma^2 K_n^3}{n} \right)^{1/4} \leq C\gamma_n(\delta).$$

Therefore,

$$\begin{aligned} \mathbb{E}[1_B(\tilde{W}_n(x_1^n))] &\leq \mathbb{E}[1_{B^{\psi_n}}(W(x_1^n)(\tau))] + 2/n \\ &\leq \mathbb{E}[f(W(x_1^n)(\tau))]/(1 - \varepsilon) + 2/n \\ &\leq \mathbb{E}[f(W^0(\tau))]/(1 - \varepsilon) + C\gamma_n(\delta) \\ &\leq \mathbb{E}[1_{B^{\psi_n+3\delta}}(W^0(\tau))] + C\gamma_n(\delta) \\ &\leq \mathbb{E}[1_{B^{2\psi_n+3\delta}}(\tilde{W}^0)] + C\gamma_n(\delta), \end{aligned}$$

where C is varying from line to line. The claim of the theorem follows by applying Theorem A.2. \square

APPENDIX D. PROOFS FOR SECTION 4

Proof of Proposition 4.1. Pick any $f \in \mathcal{F}$. By the triangle inequality, we have for any $x \in \mathcal{X}$,

$$\frac{\sqrt{n}|\hat{f}_n(x, \hat{l}_n) - f(x)|}{\hat{\sigma}_n(x, \hat{l}_n)} \leq |Z_{n,f}(x, \hat{l}_n)| \frac{\sigma_{n,f}(x, \hat{l}_n)}{\hat{\sigma}_n(x, \hat{l}_n)} + \Delta_{n,f}(\hat{l}_n),$$

by which we have

$$\begin{aligned} & \mathbb{P}_f\{f(x) \in C_n(x), \forall x \in \mathcal{X}\} \\ & \geq \mathbb{P}_f \left\{ |Z_{n,f}(x, \hat{l}_n)| \frac{\sigma_{n,f}(x, \hat{l}_n)}{\hat{\sigma}_n(x, \hat{l}_n)} + \Delta_{n,f}(\hat{l}_n) \leq \hat{c}_n(\alpha) + c'_n, \forall x \in \mathcal{X} \right\}. \quad (\text{D.1}) \end{aligned}$$

By Condition H4, $\mathbb{P}_f\{\Delta_{n,f}(\hat{l}_n) > c'_n\} \leq \delta_{4n}$, and so the probability in D.1 is bounded from below by

$$\mathbb{P}_f \left(\sup_{x \in \mathcal{X}} \frac{|\hat{f}_n(x, \hat{l}_n) - \mathbb{E}_f[\hat{f}_n(x, l)]|_{l=\hat{l}_n}}{\hat{\sigma}_n(x, \hat{l}_n)} \leq \hat{c}_n(\alpha), \forall x \in \mathcal{X} \right) - \delta_{4n},$$

which is in turn bounded from below by

$$\begin{aligned} & \mathbb{P}_f \left(\sup_{v \in \mathcal{V}_n} \frac{|\hat{f}_n(v) - \mathbb{E}_f[\hat{f}_n(v)]|}{\hat{\sigma}_n(v)} \leq \hat{c}_n(\alpha), \forall x \in \mathcal{X} \right) - \delta_{4n} \\ & \geq 1 - \alpha - \tau_n - \delta_n - A\bar{\epsilon}_{n,f}\mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}] \end{aligned}$$

where the second line follows from Lemma D.1. Combining inequalities above completes the proof of the first claim.

To prove the second claim, note that $\delta_n \leq Cn^{-c}$ and $\tau_n \leq Cn^{-c}$ by conditions H1-H4. Further, by Markov's inequality, $c_{n,f}(\alpha) \leq \mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}]/\alpha \leq C\sqrt{\log n}$, and so $\bar{\epsilon}_{n,f} \leq Cn^{-c}$. Therefore, the second claim follows from the first claim. \square

Proof of Proposition 4.2. By construction,

$$\sup_{x \in \mathcal{X}} \lambda(C_n(x)) = 2(\hat{c}_n(\alpha) + c'_n) \frac{\sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, \hat{l}_n)}{n^{1/2}}.$$

Therefore, (4.3) follows from Conditions H2, H5, and H6.

We now prove (4.4). Since τ_n and ϵ_{2n} are both bounded by $C_1 n^{-c_1}$ (condition H2), there exists n_0 such that $\tau_n \leq \alpha/2$ and $\epsilon_{2n} \leq 1$ for $n \geq n_0$. For $n < n_0$, (4.4) holds by choosing sufficiently large C . Consider $n \geq n_0$. Then

$$c_{n,f}(\alpha - \tau_n) + \epsilon_{2n} \leq c_{n,f}(\alpha/2) + 1$$

By Theorem A.5, $c_{n,f}(\alpha/2) \leq \mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}] + \sqrt{2|\log(\alpha/2)|}$. By Condition H1, $\mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}] \geq 1$, and so $c_{n,f}(\alpha/2) + 1 \leq C\mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}]$ for some constant C depending on α only. Further, combining $\mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}] \geq 1$ and $u_n \geq c_1$ (Condition H5) gives

$$\bar{c}_n \leq Lu_n \sqrt{\log n}$$

where L depends on α , c_1 and C_1 only. Substituting this expression into (4.3) yields (4.4), which concludes the proof of the proposition. \square

Lemma D.1. *Let Conditions H1-H3 hold. Then*

$$\begin{aligned} \mathbb{P}_f \left(\sup_{v \in \mathcal{V}_n} \frac{|\hat{f}_n(v) - \mathbb{E}_f[\hat{f}_n(v)]|}{\hat{\sigma}_n(v)} > \hat{c}_n(\alpha) \right) \\ \leq \alpha + \tau_n + \delta_{1n} + \delta_{2n} + \delta_{3n} + A(\epsilon_{1n} + \epsilon_{2n} + \epsilon_{3n} c_{n,f}(\alpha)) \mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}] \end{aligned}$$

for some absolute constant A .

Proof of Lemma D.1. Recall that $W_{n,f} = \|Z_{n,f}\|_{\mathcal{V}_n}$. Using Conditions H2 and H3 shows that probability in the statement of the lemma is bounded from above by

$$\begin{aligned} \mathbb{P}_f \left(\sup_{v \in \mathcal{V}_n} \frac{|\hat{f}_n(v) - \mathbb{E}_f[\hat{f}_n(v)]|}{\hat{\sigma}_n(v)} \leq \hat{c}_n(\alpha) \right) \\ \geq \mathbb{P}_f(W_{n,f} \leq (1 - \epsilon_{3n})c_{n,f}(\alpha + \tau_n) - \epsilon_{1n} - \epsilon_{2n} - \delta_{2n} - \delta_{3n}). \end{aligned} \quad (\text{D.2})$$

Using the Gaussian approximation assumed in Condition H1, we now have

$$(D.2) = \mathbb{P}_f \{ W_{n,f}^0 \leq (1 - \epsilon_{3n})c_{n,f}(\alpha + \tau_n) - \epsilon_{1n} - \epsilon_{2n} - \delta_{1n} - \delta_{2n} - \delta_{3n} \}.$$

Recalling the definition of the concentration function, the probability in the expression above is bounded from below by

$$\begin{aligned} \mathbb{P}_f \{ W_{n,f}^0 \leq c_{n,f}(\alpha + \tau_n) - \bar{\epsilon}_{n,f} \} &\geq \mathbb{P}_f \{ W_{n,f}^0 \leq c_{n,f}(\alpha + \tau_n) \} - p_{\bar{\epsilon}_{n,f}}(|G_{n,f}|) \\ &\geq 1 - \alpha - \tau_n - p_{\bar{\epsilon}_{n,f}}(|G_{n,f}|). \end{aligned}$$

Applying Corollary 2.1 to bound $p_{\bar{\epsilon}_{n,f}}(|G_{n,f}|)$ gives the asserted claim. \square

APPENDIX E. PROOFS FOR SECTION 5

Proof of Proposition 5.1. There exists n_0 such that $c_2 \geq C_5 n^{-c_5}$ for all $n \geq n_0$. For $n < n_0$, set $\delta_{1n} = \delta_{2n} = \delta_{3n} = 1$. Then Conditions H1-H3 hold for these n 's by choosing sufficiently large C_1 and sufficiently small c_1 . Consider $n \geq n_0$. Condition H1 follows from Theorem 3.1 and Corollary 2.2.8 in [27]. Consider condition H3. Note that

$$\left| \frac{\hat{\sigma}_n(x, l)}{\sigma_{n,f}(x, l)} - 1 \right| \leq \left| \frac{\hat{\sigma}_n^2(x, l)}{\sigma_{n,f}^2(x, l)} - 1 \right|. \quad (\text{E.1})$$

Define $\mathcal{K}_{n,f}^2 := \{g^2 : g \in \mathcal{K}_{n,f}\}$. Given the definition of $\hat{\sigma}_n(x, l)$, the RHS of (E.1) is bounded by

$$\sup_{g \in \mathcal{K}_{n,f}^2} |\mathbb{E}_n[g(X_i)] - \mathbb{E}[g(X_1)]| + \sup_{g \in \mathcal{K}_{n,f}} |\mathbb{E}_n[g(X_i)]^2 - \mathbb{E}[g(X_1)]^2|. \quad (\text{E.2})$$

It is easy to check that the function class $\mathcal{K}_{n,f}^2$ is VC type with constants $2A$ and V and constant envelope b_n^2 . Moreover, for all $g \in \mathcal{K}_{n,f}^2$,

$$\mathbb{E}[g(X_i)^2] \leq b_n^2 \mathbb{E}[g(X_i)] \leq b_n^2 \sigma_n^2.$$

Therefore, Talagrand's inequality (Theorem A.4) with $t = \log n$, which can be applied because $b_n^2 K_n/n \leq b_n^2 K_n^4/n \leq C_5 n^{-c_5} \leq c_2 \leq \sigma_n^2$, gives

$$P \left(\sup_{g \in \mathcal{K}_{n,f}^2} |\mathbb{E}_n[g(X_i)] - \mathbb{E}[g(X_1)]| > \frac{1}{2} \sqrt{\frac{b_n^2 \sigma_n^2 K_n}{n}} \right) \leq \frac{1}{n}. \quad (\text{E.3})$$

In addition,

$$\sup_{g \in \mathcal{K}_{n,f}} |\mathbb{E}_n[g(X_i)]^2 - \mathbb{E}[g(X_1)]^2| \leq 2b_n \sup_{g \in \mathcal{K}_{n,f}} |\mathbb{E}_n[g(X_i)] - \mathbb{E}[g(X_1)]|,$$

and so another application of Talagrand's inequality yields

$$P \left(\sup_{g \in \mathcal{K}_{n,f}} |\mathbb{E}_n[g(X_i)]^2 - \mathbb{E}[g(X_1)]^2| > \frac{1}{2} \sqrt{\frac{b_n^2 \sigma_n^2 K_n}{n}} \right) \leq \frac{1}{n}. \quad (\text{E.4})$$

Given that $b_n^2 \sigma_n^2 K_n/n \leq Cn^{-c}$ for some $c, C > 0$, combining (E.1)-(E.4) gives condition H3 with $\epsilon_{3n} := (b_n^2 \sigma_n^2 K_n/n)^{1/2}$ and $\delta_{3n} := 2/n$.

Finally, we verify condition H2. Define

$$\tilde{\mathbb{G}}_n(x, l) = \tilde{\mathbb{G}}_n(X_1^n, \xi_1^n)(x, l) := \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \frac{K_l(X_i - x) - \hat{f}_n(x, l)}{\sigma_n(x, l)}.$$

and

$$\Delta \mathbb{G}_n(x, l) = \hat{\mathbb{G}}_n(x, l) - \tilde{\mathbb{G}}_n(x, l).$$

In addition, for all $x_1^n \in \mathbb{R}^{nd}$, define

$$\begin{aligned} \hat{W}_n(x_1^n) &:= \sup_{(x,l) \in \mathcal{X} \times \mathcal{L}_n} \hat{\mathbb{G}}_n(x_1^n, \xi_1^n)(x, l), \\ \tilde{W}_n(x_1^n) &:= \sup_{(x,l) \in \mathcal{X} \times \mathcal{L}_n} \tilde{\mathbb{G}}_n(x_1^n, \xi_1^n)(x, l). \end{aligned}$$

Consider the set $S_{n,1} \subset \mathbb{R}^{nd}$ of values X_1^n such that whenever $X_1^n \in S_{n,1}$, $|\hat{\sigma}_n(x, l)/\sigma_{n,f}(x, l) - 1| \leq \epsilon_{3n}$ for all $(x, l) \in \mathcal{X} \times \mathcal{L}_n$. Calculations above show that $P(S_{n,1}) \geq 1 - \delta_{3n} = 1 - 2/n$. Pick any $x_1^n \in S_{n,1}$. Then

$$\Delta \mathbb{G}_n(x_1^n, \xi_1^n)(x, l) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i \frac{K_l(x_i - x) - \hat{f}_n(x, l)}{\sigma_n(x, l)} \left(\frac{\sigma_n(x, l)}{\hat{\sigma}_n(x, l)} - 1 \right)$$

is a zero-mean Gaussian process with

$$\text{var}(\Delta \mathbb{G}_n(x_1^n, \xi_1^n)(x, l)) = \frac{\hat{\sigma}_n^2(x, l)}{\sigma_n^2(x, l)} \left(\frac{\sigma_n(x, l)}{\hat{\sigma}_n(x, l)} - 1 \right)^2 \leq 9\epsilon_{3n}^2$$

whenever $\epsilon_{3n} \leq 1/2$. From now on, we will assume that n is sufficiently large so that $\epsilon_{3n} \leq 1/2$. Further, let $\tilde{\mathcal{K}}_{n,f} := \{ag : a \in (0, 1], g \in \mathcal{K}_{n,f}\}$. It is easy to check that $\tilde{\mathcal{K}}_{n,f}$ is VC type with constants $2A$ and $2V$ and constant envelope b_n . It is also easy to check that uniform covering numbers of the process $\Delta \mathbb{G}_n(x_1^n, \xi_1^n)$ with respect to the natural (standard deviation)

semimetric are bounded by uniform covering numbers of the function class $\tilde{\mathcal{K}}_{n,f}$. So, applying Theorem A.2.7 in [27] gives for all $\lambda \geq K_n^{1/2} \epsilon_{3n}$,

$$\begin{aligned} \mathbb{P}\left(|\hat{W}_n(x_1^n) - \tilde{W}_n(x_1^n)| \geq \lambda\right) &\leq \mathbb{P}\left(\sup_{(x,l) \in \mathcal{X} \times \mathcal{L}_n} |\Delta \mathbb{G}_n(x_1^n, \xi_1^n)(x, l)| \geq \lambda\right) \\ &\leq \exp(K_n - \lambda^2/18\epsilon_{3n}^2). \end{aligned}$$

Recall that $\epsilon_{3n}^2 = b_n^2 \sigma_n^2 K_n/n$. Since $K_n \geq C \log n$ and $b_n^2 \sigma_n^2 K_n^2/n \leq Cn^{-c}$, it follows that there exists λ such that

$$\lambda \leq Cn^{-c} \text{ and } \mathbb{P}(|\hat{W}_n(x_1^n) - \tilde{W}_n(x_1^n)| \geq \lambda) \leq Cn^{-c} \quad (\text{E.5})$$

uniformly over $x_1^n \in \mathbb{R}^{nd}$. By Theorem 3.2 and condition $b_n^2 K_n^4/n \leq C_1 n^{-c_1}$, there exists a measurable set $S_{n,2} \subset \mathbb{R}^{nd}$ such that $\mathbb{P}(S_{n,2}) \geq 1 - 3/n$ and for any $x_1^n \in S_{n,0}$ one can construct a random variable W^0 such that $W^0 \stackrel{d}{=} \|G_{n,f}\|_{\mathcal{V}_n}$ and there exists (possibly different) λ such that

$$\lambda \leq Cn^{-c} \text{ and } \mathbb{P}(|\tilde{W}_n(x_1^n) - W^0| \geq \lambda) \leq Cn^{-c} \quad (\text{E.6})$$

uniformly over $x_1^n \in S_{n,2}$. Combining (E.5) and (E.6) shows that uniformly over $x_1^n \in S_{n,0} := S_{n,1} \cap S_{n,2}$,

$$\mathbb{P}(|\hat{W}_n(x_1^n) - W^0| \geq \lambda) \leq Cn^{-c}$$

for some $\lambda \leq Cn^{-c}$.

Let $\hat{c}_n(\alpha, x_1^n)$ be conditional $(1 - \alpha)$ -quantile of $\|\hat{G}_n(x_1^n, \xi_1^n)\|_{\mathcal{V}_n}$. Then $\hat{c}_n(\alpha) = \hat{c}_n(\alpha, X_1^n)$ and for any $x_1^n \in S_{n,0}$, we have

$$\begin{aligned} &\mathbb{P}\{\|G_{n,f}\|_{\mathcal{V}_n} \leq \hat{c}_n(\alpha, x_1^n) + \lambda\} \\ &= \mathbb{P}\{W^0 \leq \hat{c}_n(\alpha, x_1^n) + \lambda\} \\ &\geq \mathbb{P}\{W^0 \leq \hat{c}_n(\alpha, x_1^n) + \lambda\} \cap \{|\hat{W}_n(x_1^n) - W^0| \leq \lambda\} \\ &\geq \mathbb{P}\{\hat{W}_n(x_1^n) \leq \hat{c}_n(\alpha, x_1^n)\} - Cn^{-c} \\ &\geq 1 - \alpha - Cn^{-c}, \end{aligned}$$

by which we have $\hat{c}_n(\alpha) \geq c_{n,f}(\alpha + Cn^{-c}) - \lambda$ whenever $X_1^n \in S_{n,0}$, which happens with probability at least $1 - 5/n$. This completes the proof of part (a) of condition H2. Part (b) of condition H2 follows similarly. \square

Proof of Proposition 5.2. In the proof of this proposition, we will assume that Conditions H1-H3 hold. Moreover, we will assume that Condition H2 holds uniformly over all $\alpha \in (0, 1)$. Indeed, these conditions are verified in Proposition 5.1 under weaker assumptions than those imposed here.

Fix $f \in \mathcal{F}$. By condition L3, there exists $t \in [c_3, C_3]$ such that equation (5.5) holds with these f and t . In fact, it is easy to see that t is defined uniquely. This defines the function $t : \mathcal{F} \rightarrow \mathbb{R}$ appearing in Condition H6.

By Condition H3, ϵ_{3n} is bounded by $C_1 n^{-c_1}$. So, there exists n_0 such that $\epsilon_{3n} \leq 1/2$ for all $n \geq n_0$. Let $\delta_{4n} = \delta_{6n} = 1$ for $n < n_0$, so that Conditions

H4 and H6 hold these n 's with C_1 sufficiently large and c_1 sufficiently small. Consider $n \geq n_0$.

Let $m > s$ be such that $c_4 2^{(m-s)c_3} > C_4$, $M_1 > 0$ be such that $M_1(1 - C_4/(c_4 2^{(m-s)c_3})) > 2(q+1)C_4/c_4$, and $M_2 > 0$ be such that $M_2 < (4/9)(q-1)(c_2/C_2)$. For $M > 0$, define

$$l^*(M) := \inf \left\{ l \in \mathcal{L}_n : C_4 2^{-lt} \sqrt{n} \leq M \hat{c}(\gamma_n) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) \right\},$$

and let $l_1^* := l^*(M_1)$ and $l_2^* := l^*(M_2)$. We will invoke the following lemmas.

Lemma E.1. $\lambda_1 := P_f(\hat{l}_n < l_1^* - m) \leq Cn^{-c}$ uniformly over $f \in \mathcal{F}$.

Proof of Lemma E.1. Define $\mathcal{L}_n^m := \{l \in \mathcal{L}_n : l < l_1^* - m\}$. If there is no $l' \in \mathcal{L}_n$ such that $l' < l_1^*$, we are done. Otherwise, since $l_1^* \in \mathcal{L}_n$ by the fact that \mathcal{L}_n is closed, there exists some $l' \in \mathcal{L}_n$ such that $l' \in (l_1^* - s, l_1^*)$. Fix this l' . Then

$$\begin{aligned} P_f(\hat{l}_n < l_1^* - m) &\leq P_f \left(\inf_{l \in \mathcal{L}_n^m} \sup_{x \in \mathcal{X}} \frac{\sqrt{n} |\hat{f}_n(x, l) - \hat{f}_n(x, l')|}{\hat{\sigma}_n(x, l) + \hat{\sigma}_n(x, l')} \leq q \hat{c}_n(\gamma_n) \right) \\ &\leq P_f \left(\inf_{l \in \mathcal{L}_n^m} \frac{\sup_{x \in \mathcal{X}} \sqrt{n} |\hat{f}_n(x, l) - \hat{f}_n(x, l')|}{\sup_{x \in \mathcal{X}} (\hat{\sigma}_n(x, l) + \hat{\sigma}_n(x, l'))} \leq q \hat{c}_n(\gamma_n) \right). \end{aligned}$$

By triangle inequality,

$$\begin{aligned} \sup_{x \in \mathcal{X}} |\hat{f}_n(x, l) - \hat{f}_n(x, l')| &\geq c_4 2^{-lt} - C_4 2^{-l't} \\ &\quad - \sup_{x \in \mathcal{X}} |\hat{f}_n(x, l) - E_f[\hat{f}_n(x, l)]| - \sup_{x \in \mathcal{X}} |\hat{f}_n(x, l') - E_f[\hat{f}_n(x, l')]|. \end{aligned}$$

Further, for $l \in \mathcal{L}_n^m$, by the definition of l^* , construction of M_1 , and since $t \geq c_3$,

$$\begin{aligned} \frac{c_4 2^{-lt} - C_4 2^{-l't}}{2} &\geq \frac{c_4}{2^{lt+1}} \left(1 - \frac{C_4}{c_4 2^{(m-s)c_3}} \right) \\ &\geq \frac{c_4}{2C_4 \sqrt{n}} M \hat{c}(\gamma_n) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) \left(1 - \frac{C_4}{c_4 2^{(m-s)c_3}} \right) \\ &\geq (q+1) \hat{c}_n(\gamma_n) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) / \sqrt{n} \end{aligned}$$

and

$$\begin{aligned} \frac{c_4 2^{-lt} - C_4 2^{-l't}}{2} &\geq \frac{C_4}{2^{l't+1}} \left(\frac{c_4 2^{(m-s)c_3}}{C_4} - 1 \right) \\ &\geq \frac{1}{2\sqrt{n}} M \hat{c}(\gamma_n) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l') \left(\frac{c_4 2^{(m-s)c_3}}{C_4} - 1 \right) \\ &\geq (q+1) \hat{c}_n(\gamma_n) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l') / \sqrt{n}. \end{aligned}$$

Combining these inequalities yields

$$\begin{aligned}
& \mathbb{P}_f(\hat{l}_n < l_1^* - m) \\
& \leq \mathbb{P}_f \left(\sup_{l \in \mathcal{L}_n^m} \left[\sup_{x \in \mathcal{X}} \sqrt{n} |\hat{f}_n(x, l) - \mathbb{E}_f[\hat{f}_n(x, l)]| - \hat{c}_n(\gamma_n) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) \right] \geq 0 \right) \\
& + \mathbb{P}_f \left(\sup_{x \in \mathcal{X}} \sqrt{n} |\hat{f}_n(x, l') - \mathbb{E}_f[\hat{f}_n(x, l')]| - \hat{c}_n(\gamma_n) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l') \geq 0 \right) \\
& \leq 2\mathbb{P}_f \left(\sup_{v \in \mathcal{V}_n} \frac{\sqrt{n} |\hat{f}_n(v) - \mathbb{E}[\hat{f}_n(v)]|}{\hat{\sigma}_n(v)} \geq \hat{c}_n(\gamma_n) \right) \leq 2\gamma_n + Cn^{-c} \leq Cn^{-c}
\end{aligned}$$

where the inequality preceding the last one follows from Lemma D.1 because $c_{n,f}(\gamma_n) \leq C\sqrt{\log n}$, which in turn follows from Theorem A.5 and the facts that $|\log \gamma_n| \leq C\sqrt{\log n}$ and $\mathbb{E}[\|G_{n,f}\|_{\mathcal{V}_n}] \leq C\sqrt{\log n}$. This gives the asserted claim. \square

Lemma E.2. $\lambda_2 := \mathbb{P}_f(\hat{l}_n > l_2^*) \leq Cn^{-c}$ uniformly over $f \in \mathcal{F}$.

Proof of Lemma E.2. Note that with probability at least $1 - \delta_{3n}$, for all $l \in \mathcal{L}_n$,

$$\sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) \leq (1 + \epsilon_{3n}) \sup_{x \in \mathcal{X}} \sigma_{n,f}(x, l) \leq (1 + \epsilon_{3n}) C_2 2^{ld/2}.$$

Therefore, with the same probability,

$$C_4 2^{-l_2^* t} \sqrt{n} \leq M_2 \hat{c}_n(\gamma_n) (1 + \epsilon_{3n}) C_2 2^{l_2^* d/2},$$

and so for all $l \geq l_2^*$,

$$\begin{aligned}
C_4 2^{-lt} \sqrt{n} & \leq M_2 \hat{c}_n(\gamma_n) (1 + \epsilon_{3n}) C_2 2^{ld/2} \\
& \leq M_2 \hat{c}_n(\gamma_n) (1 + \epsilon_{3n}) (C_2/c_2) \inf_{x \in \mathcal{X}} \sigma_{n,f}(x, l) \\
& \leq M_2 \hat{c}_n(\gamma_n) (1 + \epsilon_{3n})^2 (C_2/c_2) \inf_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) \\
& \leq (q - 1) \hat{c}_n(\gamma_n) \inf_{x \in \mathcal{X}} \hat{\sigma}_n(x, l)
\end{aligned}$$

where the last inequality follows from the choice of M_2 and the fact that $(1 + \epsilon_{3n})^2 \leq 9/4$ for $n \geq n_0$. So,

$$\mathbb{P}_f \left(\sup_{l \geq l_2^*} \Delta_{n,f}(l) > (q - 1) \hat{c}_n(\gamma_n) \right) \leq \delta_{3n}. \quad (\text{E.7})$$

Further, by the definition of \hat{l}_n , triangle inequality, and union bound,

$$\begin{aligned}
\mathbb{P}_f(\hat{l}_n > l_2^*) & \leq \mathbb{P}_f \left(\sup_{l, l' > l_2^*} \sup_{x \in \mathcal{X}} \frac{\sqrt{n} |\hat{f}_n(x, l) - \hat{f}_n(x, l')|}{\hat{\sigma}_n(x, l) + \hat{\sigma}_n(x, l')} > q \hat{c}_n(\gamma_n) \right) \\
& \leq 2\mathbb{P}_f \left(\sup_{l > l_2^*} \sup_{x \in \mathcal{X}} \frac{\sqrt{n} |\hat{f}_n(x, l) - f(x)|}{\hat{\sigma}_n(x, l)} > q \hat{c}_n(\gamma_n) \right)
\end{aligned}$$

Using the definition of $\Delta_{n,f}(l)$ and applying triangle inequality once again, the probability in the last expression can be further bounded from above by

$$\begin{aligned} & \mathbb{P}_f \left(\sup_{l>l_2^*} \sup_{x \in \mathcal{X}} \frac{\sqrt{n}|\hat{f}_n(x, l) - \mathbb{E}_f[\hat{f}_n(x, l)]|}{\hat{\sigma}_n(x, l)} + \Delta_{n,f}(l) > q\hat{c}_n(\gamma_n) \right) \\ & \leq_{(1)} \mathbb{P}_f \left(\sup_{v \in \mathcal{V}_n} \frac{\sqrt{n}|\hat{f}_n(v) - \mathbb{E}_f[\hat{f}_n(v)]|}{\hat{\sigma}_n(v)} > \hat{c}_n(\gamma_n) \right) + \delta_{3n} \\ & \leq_{(2)} \gamma_n + \delta_{3n} + Cn^{-c} \leq_{(3)} Cn^{-c} \end{aligned}$$

where (1) follows from (E.7), (2) follows from the same arguments as those in the proof of Lemma E.1, and (3) holds because γ_n and δ_{3n} are both bounded by Cn^{-c} . This gives the asserted claim. \square

Now we verify Condition H4. Note that for all $f \in \mathcal{F}$ and $l \in \mathcal{L}_n$,

$$\hat{c}_n(\gamma_n) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) \leq \hat{c}_n(\gamma_n)(1 + \epsilon_{3n}) \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) \leq C\hat{c}_n(\gamma_n)2^{ld/2}$$

with probability at least $1 - \delta_{3n}$ where the first inequality follows from Condition H3 and the second inequality holds by Condition L2. Therefore, with the same probability, l^* satisfies

$$\sqrt{n}2^{-l^*(t+d/2)} \leq C\hat{c}_n(\gamma_n). \quad (\text{E.8})$$

Hence, with probability at least $1 - \delta_{3n} - \lambda_1$,

$$\begin{aligned} \Delta_{n,f}(\hat{l}_n) & \leq_{(1)} \frac{\sqrt{n}C_42^{-\hat{l}_nt}}{\inf_{x \in \mathcal{X}} \hat{\sigma}_n(x, \hat{l}_n)} \leq_{(2)} \frac{\sqrt{n}C_42^{-\hat{l}_nt}}{(1 - \epsilon_{3n})\inf_{x \in \mathcal{X}} \sigma_{n,f}(x, \hat{l}_n)} \\ & \leq_{(3)} \frac{\sqrt{n}C_42^{-\hat{l}_nt}}{(1 - \epsilon_{3n})c_22^{\hat{l}_nd/2}} \leq_{(4)} C\sqrt{n}2^{-\hat{l}_n(t+d/2)} \\ & \leq_{(5)} C\sqrt{n}2^{-(l^*-m)(t+d/2)} \leq_{(6)} C\hat{c}_n(\gamma_n) \end{aligned}$$

where (1) follows from Condition L3, (2) is by Condition H3, (3) is by Condition L2, (4) holds because $\epsilon_{3n} \leq 1/2$, (5) is by Lemma E.1, and (6) follows from (E.8). This completes the verification of Condition H4.

Next, we verify Condition H6. Note that with probability at least $1 - \delta_{3n}$, for all $l \in \mathcal{L}_n$,

$$\sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, l) \geq (1 - \epsilon_{3n}) \sup_{x \in \mathcal{X}} \sigma_{n,f}(x, l) \geq (1 - \epsilon_{3n})c_22^{ld/2}.$$

In addition, by construction, $\hat{c}_n(\gamma_n)$ is the $(1 - \gamma_n)$ -quantile of the maximum over a set of $N(0, 1)$ random variables. Since $\gamma_n \leq C_5n^{-c_5}$, this implies that $\hat{c}_n(\gamma_n) \geq c\sqrt{\log n}$ for some constant $c > 0$. Therefore, with probability at least $1 - \delta_{3n}$,

$$2^{-l_2^*t}\sqrt{n} \geq c\sqrt{\log n}2^{l_2^*d/2}.$$

So, by Lemma E.2, with probability at least $1 - \delta_{3n} - \lambda_2$,

$$2^{-\hat{l}_nt}\sqrt{n} \geq c\sqrt{\log n}2^{\hat{l}_nd/2}.$$

Conclude that with the same probability

$$\begin{aligned} \sup_{x \in \mathcal{X}} \hat{\sigma}_n(x, \hat{l}_n) &\leq (1 + \epsilon_{3n}) \sup_{x \in \mathcal{X}} \sigma_{n,f}(x, \hat{l}_n) \\ &\leq C_2 2^{\hat{l}_n d/2} \leq C \left(\frac{\log n}{n} \right)^{-d/(2t(f)+d)}. \end{aligned}$$

Since $\delta_{3n} + \lambda_2 \leq Cn^{-c}$, Condition H6 follows.

Finally, we verify Condition H5. Let $M(X_1^n)$ denote the median of $\hat{W}_n(x_1^n) = \|\hat{\mathbb{G}}_n(x_1^n, \xi_1^n)\|_{\mathcal{V}_n}$. Applying Theorem A.5 conditional on the data gives $\hat{c}_n(\gamma_n) \leq M(X_1^n) + \sqrt{2|\log \gamma_n|}$. Further, in the proof of Proposition 5.1, it was shown that there exists a measurable set $S_{n,0}$ in \mathbb{R}^{nd} such that $P_f(X_1^n \notin S_{n,0}) \leq Cn^{-c}$ and for each $x_1^n \in S_{n,0}$ one can construct a random variable W^0 such that $P(|\hat{W}_n(x_1^n) - W^0| \geq \zeta_1) \leq \zeta_2$ for some ζ_1 and ζ_2 both bounded by Cn^{-c} and $W_{n,f}^0 \stackrel{d}{=} \|G_{n,f}\|_{\mathcal{V}_n}$. Therefore,

$$P_f(M(X_1^n) > c_{n,f}(1/2 + \zeta_2) + \zeta_1) \leq Cn^{-c} \quad (\text{E.9})$$

with probability at least $1 - Cn^{-c}$. Since $E[\|G_{n,f}\|_{\mathcal{V}_n}] \leq C_1 \sqrt{\log n}$ (assumed in Condition H1), Markov's inequality implies that $c_{n,f}(1/2 + \zeta_2) \leq C\sqrt{\log n}$. Combining this inequality with (E.9) gives Condition H5. This completes the proof of the proposition. \square

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